

Asymptotic Generalization of Reissner-Mindlin Theory: Accurate Three-Dimensional Recovery for Composite Shells

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Abstract

A rigorous and systematic dimensional reduction of a shell-like structure is undertaken. It starts with geometrically nonlinear, three-dimensional, anisotropic elasticity theory and takes advantage of small parameters associated with the geometry. This reduction is carried out using the Variational Asymptotic Method and splits the three-dimensional problem into a linear, one-dimensional, through-the-thickness analysis and a nonlinear, two-dimensional, shell analysis. The two-dimensional equations are put into the form of a nonlinear Reissner-Mindlin shell theory, details of which are dealt with in a separate paper. The focus of this paper is on the through-the-thickness analysis, which is solved by a one-dimensional finite element method and which provides two useful pieces of information: a generalized two-dimensional constitutive law for the shell equations, and a set of recovery relations that can be used to express the three-dimensional field variables through the thickness in terms of two-dimensional shell variables calculated in the shell analysis. The resulting analysis can be incorporated into standard Reissner-Mindlin shell finite element codes. Numerical results are compared with the exact solution, and the excellent agreement validates the fidelity of this modeling approach.

Key words: Reissner-Mindlin, composite, shell, variational asymptotic method

1 Introduction

Although composite materials have found increasing applications in engineering due to their superior engineering properties and enhanced manufacturing technology, their application is not so extensive as one might expect. One reason is that the old tools used for designing structures made of isotropic

materials are inadequate for use with composites, and the analysis of composite structures is much more complicated than that for isotropic structures. Although many new models have appeared in the literature, design engineers have been reluctant to accept them with confidence. This is partly because many new models are constructed for specific problems without generalization in mind and partly because some models are too complicated and computationally inefficient to be used for design purposes. Simple yet efficient and generalized methods of analysis are still needed to shorten the design period and reduce the cost of composite structures.

Many engineering structures made with composite materials have one dimension much smaller than the other two and can be modeled as shells. Shell models are generally derived from three-dimensional (3-D) elasticity theory, making use of the fact that the shell is thin in some sense. The simplest composite shell theory is the classical lamination theory, which is based on the Kirchhoff hypothesis. It is well known, however, that composite shells do not have to be very thick in order for this theory to yield extremely poor results compared to the actual 3-D solution.

Although it is plausible to take into account the smallness of the thickness of such structures, construction of an accurate two-dimensional (2-D) model for a 3-D body still introduces a lot of challenges. There have been many attempts to rationally improve upon the classical model, almost all of which have serious shortcomings. One will appreciate this by reading recent review papers [9,8]. Most of the models that have appeared in the literature [10,13,19,3] are based on *ad hoc* kinematic assumptions that cannot be reasonably justified for composite structures, such as an *a priori* distribution of displacement through the thickness.

From a mathematical point of view, the approximation in this dimensional reduction process stems from elimination of the thickness coordinate from the independent variables of the governing partial differential equations of equilibrium. This sort of approximation is inevitable if one wants to take advantage of the smallness of the thickness to simplify the analysis. However, other approximations that are not absolutely necessary should be avoided. For example, for small-strain analysis of shells, it is reasonable to assume that the thickness, h , is small compared to the wavelength of deformation of the reference surface, l . However, it is not at all reasonable to assume *a priori* some *ad hoc* displacement field, although that is the way most existing shell theories have been constructed.

In this paper we will proceed in a very different manner. We first cast the original 3-D elasticity problem in a form that introduces 2-D intrinsic variables for the shell. This can be done in such a way as to be applicable for arbitrarily large displacement and global rotation, subject only to the strain

being small [4,6]. Then, a systematic approach can be employed to reduce the dimensionality in terms of the smallness of h/R , where R is the minimum radius of curvature for the shell structure, and h/l . The present work uses the Variational Asymptotic Method (VAM) [2] to split the original nonlinear 3-D elasticity problem into a linear, one-dimensional (1-D), normal-line analysis and a nonlinear, 2-D, shell analysis.

The normal-line analysis produces a constitutive model to be used in the 2-D shell analysis, along with recovery relations that yield the 3-D fields for displacement, strain and stress using results obtained from the solution of the 2-D problem. The resulting shell theory is of the Reissner-Mindlin type but is geometrically exact. It is noted that shear deformable shell theories are said to be of the Reissner-Mindlin type if the only generalized strains in the analysis of the shell reference surface are three in-plane membrane strains, three out-of-plane curvature strains and two transverse shear strains. This paper does not focus on the resulting 2-D theory, however, but on the through-the-thickness analysis. For this reason, there is no need to review here the extensive collection of published papers on shell theories. A detailed exposition of the 2-D theory implied by the dimensional reduction herein can be found in [17].

The work is an extension of the previous work on plates by the authors [18], which is an extension of the pioneering work of [15,14]. The derivation has been greatly modified to accommodate the complexity of shell geometry, finally yielding a geometrical correction of order h/R to the energy. Moreover, the present approach differs the work of Sutyryn [15,14] at least in the following three aspects. First, the theory introduced in these earlier works is restricted to be linear, while the present formulation is in an intrinsic form which is good for geometrically-exact nonlinear analysis. Second, a general form of warping field is assumed *a priori* in the earlier works, and the higher-order warping is used as a parameter to solve for the assumed functions. However, in the present work the warping field is solved by usual procedure of calculus of variations. Third, the earlier works only treat composite plates, and the mathematical complexity of the approach used in those works defies extension to treat composite shells in a similar fashion.

The present theory has been implemented into the computer program – Variational Asymptotic Plate and Shell Analysis (VAPAS), a 1-D finite element code. VAPAS can be incorporated into standard 2-D shell solvers to provide an efficient and accurate analysis of composite shells, including the recovery of all 3-D field variables.

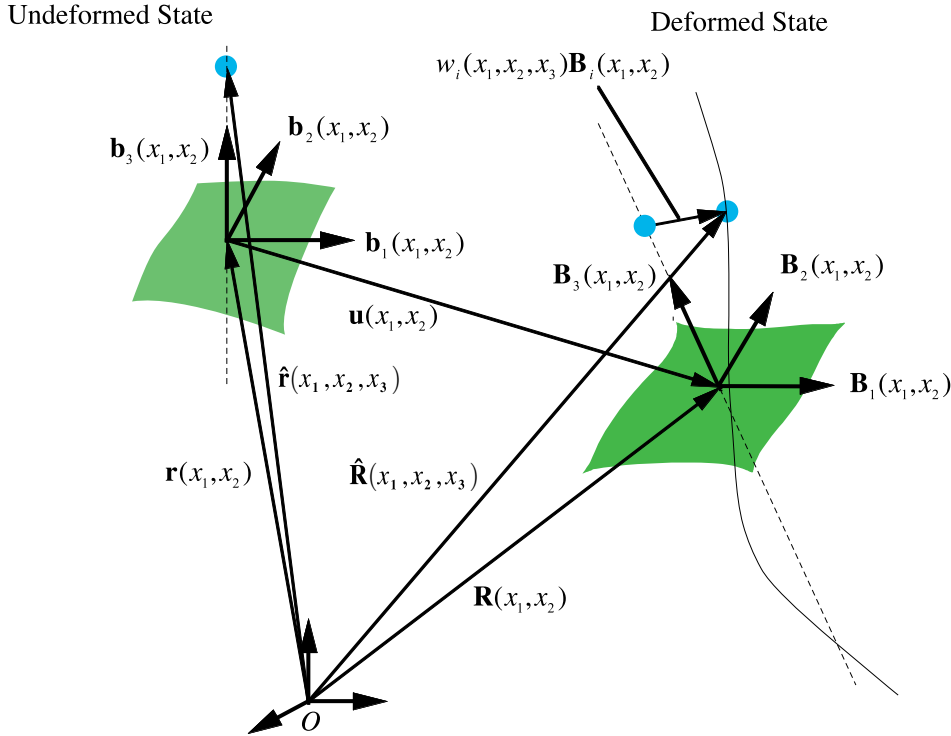


Fig. 1. Schematic of shell deformation

2 3-D Formulation

A shell is a 3-D body with a relatively small thickness h and a smooth reference surface usually chosen to be the mid-surface (see Fig. 1). The geometry of the reference surface can be mathematically represented by a set of arbitrary curvilinear coordinates, x_α . (Here and throughout the paper, Greek indices assume values 1 and 2 while Latin indices assume 1, 2, and 3. Dummy indices are summed over their range except where explicitly indicated.) However, without loss of generality, one could choose the lines of curvatures to be the curvilinear coordinates to simplify the formulation. In fact, shell theories are almost exclusively based on this choice. One has to specify another coordinate outside the reference surface to describe a 3-D medium uniquely. It is a natural and convenient choice to take the third coordinate x_3 as the normal coordinate. Letting $\mathbf{b}_3(x_1, x_2)$ denote the unit vector normal to the reference surface, one can then describe the position of any material point in the undeformed configuration by its position vector $\hat{\mathbf{r}}$ from a fixed point O , such that

$$\hat{\mathbf{r}}(x_1, x_2, x_3) = \mathbf{r}(x_1, x_2) + x_3\mathbf{b}_3(x_1, x_2) \quad (1)$$

where \mathbf{r} is the position vector from O to the point located by x_α on the reference surface. When the reference surface of the undeformed shell coincides

with its middle surface, it naturally follows that

$$\langle \hat{\mathbf{r}}(x_1, x_2, x_3) \rangle = \mathbf{r}(x_1, x_2) \quad (2)$$

where the angle-brackets denote the definite integral through the thickness of the shell and will be used throughout the rest of the development.

The 2-D base vectors associated with x_α are defined as:

$$\mathbf{a}_\alpha(x_1, x_2) = \mathbf{r}_{,\alpha} \quad (3)$$

where $(\)_{,\alpha} = \frac{\partial(\)}{\partial x_\alpha}$. From henceforth, for simplicity, we will avoid explicitly including the arguments of functions unless it is not obvious for the reader to determine what they are. From Eq. (3) one can define the so-called Lamé parameters as:

$$A_\alpha(x_1, x_2) = \sqrt{\mathbf{a}_\alpha \cdot \mathbf{a}_\alpha} \quad (4)$$

Note here the summation convention is not applied here because α is not a dummy index; this rule will apply in similar situations throughout the rest of development. Then the unit vectors along coordinates x_α can be obtained as:

$$\mathbf{b}_\alpha(x_1, x_2) = \frac{\mathbf{a}_\alpha}{A_\alpha} \quad (5)$$

It is obvious that the unit vectors \mathbf{b}_i form an orthogonal triad, such that

$$\mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|} \quad (6)$$

By the definition of Eq. (3), one can get the 3-D base vectors $\mathbf{g}_i = \hat{\mathbf{r}}_{,i}$ for the chosen coordinate system:

$$\begin{aligned} \mathbf{g}_1 &= \mathbf{a}_1 + x_3 \mathbf{b}_{3,1} \\ \mathbf{g}_2 &= \mathbf{a}_2 + x_3 \mathbf{b}_{3,2} \\ \mathbf{g}_3 &= \mathbf{b}_3 \end{aligned} \quad (7)$$

From the differential geometry of surface and following Refs. [11] and [6] one can express the derivatives of unit vectors $\mathbf{b}_{i,\alpha}$ as follows:

$$\mathbf{b}_{i,\alpha} = A_\alpha(-k_{\alpha 2} \mathbf{b}_1 + k_{\alpha 1} \mathbf{b}_2 + k_{\alpha 3} \mathbf{b}_3) \times \mathbf{b}_i \quad (8)$$

which can also be written explicitly in matrix form as

$$\begin{cases} \mathbf{b}_{1,1} \\ \mathbf{b}_{2,1} \\ \mathbf{b}_{3,1} \end{cases} = A_1 \begin{bmatrix} 0 & k_{13} & -k_{11} \\ -k_{13} & 0 & -k_{12} \\ k_{11} & k_{12} & 0 \end{bmatrix} \begin{cases} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{cases} \\ \begin{cases} \mathbf{b}_{1,2} \\ \mathbf{b}_{2,2} \\ \mathbf{b}_{3,2} \end{cases} = A_2 \begin{bmatrix} 0 & k_{23} & -k_{21} \\ -k_{23} & 0 & -k_{22} \\ k_{21} & k_{22} & 0 \end{bmatrix} \begin{cases} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{cases} \quad (9)$$

where $k_{\alpha\beta}$ refers to the usual out-of-plane curvatures and $k_{12} = k_{21} = 0$ if the coordinates are chosen to be the lines of curvatures. The geodesic curvatures $k_{\alpha 3}$ do not in general vanish for the chosen coordinate system, and they can be expressed in terms of Lamé parameters as

$$k_{13} = -\frac{A_{1,2}}{A_1 A_2} \quad k_{23} = \frac{A_{2,1}}{A_1 A_2} \quad (10)$$

Using Eq. (9), one can rewrite the expression for the 3-D base vectors from Eq. (7) as

$$\begin{aligned} \mathbf{g}_1 &= A_1(1 + x_3 k_{11}) \mathbf{b}_1 \\ \mathbf{g}_2 &= A_2(1 + x_3 k_{22}) \mathbf{b}_2 \\ \mathbf{g}_3 &= \mathbf{b}_3 \end{aligned} \quad (11)$$

and the contravariant base vectors \mathbf{g}^i can be obtained trivially as

$$\begin{aligned} \mathbf{g}^1 &= \frac{\mathbf{b}_1}{A_1(1 + x_3 k_{11})} \\ \mathbf{g}^2 &= \frac{\mathbf{b}_2}{A_2(1 + x_3 k_{22})} \\ \mathbf{g}^3 &= \mathbf{b}_3 \end{aligned} \quad (12)$$

When the shell deforms, the particle that had position vector $\hat{\mathbf{r}}$ in the undeformed state now has position vector $\hat{\mathbf{R}}$ in the deformed shell. The latter can be uniquely determined by the deformation of the 3-D body. Similarly, another triad \mathbf{B}_i is introduced for the deformed configuration. Note that the \mathbf{B}_i unit vectors are just tools to enable one to express vectors and tensors in their component form during the derivation. They are not necessarily tangent to the coordinates of the deformed shell. The relation between \mathbf{B}_i and \mathbf{b}_i can be specified by an arbitrarily large rotation specified in terms of the matrix of direction cosines $C(x_1, x_2)$ so that

$$\mathbf{B}_i = C_{ij} \mathbf{b}_j \quad C_{ij} = \mathbf{B}_i \cdot \mathbf{b}_j \quad (13)$$

subject to the requirement that \mathbf{B}_i is coincident with \mathbf{b}_i when the structure is undeformed. Now the position vector $\hat{\mathbf{R}}$ can be represented as

$$\hat{\mathbf{R}}(x_1, x_2, x_3) = \mathbf{R}(x_1, x_2) + x_3 \mathbf{B}_3(x_1, x_2) + w_i(x_1, x_2, x_3) \mathbf{B}_i(x_1, x_2) \quad (14)$$

where w_i is the warping of the normal-line element. These quantities are not assumed, as in most shell theories. Rather, they are treated as unknown 3-D functions and will be solved for later. Eq. (14) is six times redundant because of the way warping introduced. Six constraints are needed to make the formulation unique. The redundancy can be removed by choosing appropriate definitions of \mathbf{R} and \mathbf{B}_i . One can define \mathbf{R} similarly as Eq. (2) to be the average position through the thickness, from which it follows that the warping functions must satisfy the three constraints

$$\langle w_i(x_1, x_2, x_3) \rangle = 0 \quad (15)$$

Another two constraints can be specified by taking \mathbf{B}_3 as the normal to the reference surface of the deformed shell. It should be noted that this choice has nothing to do with the Kirchhoff hypothesis. Indeed, it is only for convenience in the derivation. In the Kirchhoff assumption, no local deformation of the transverse normal is allowed. On the other hand, according to the present scheme we allow all possible deformation, classifying all deformation other than that of classical shell theory as warping, which is assumed to be small and to be solved by VAM. This assumption is valid if the strain is small and the local rotation (*i.e.* the rotation of the normal line due to warping) is of the order of the strain [5].

Based on the concept of decomposition of rotation tensor [5,4], the Jauman-Biot-Cauchy strain components for small local rotation are given by

$$\Gamma_{ij} = \frac{1}{2}(F_{ij} + F_{ji}) - \delta_{ij} \quad (16)$$

where F_{ij} is the mixed-basis component of the deformation gradient tensor such that

$$F_{ij} = \mathbf{B}_i \cdot \mathbf{G}_k \mathbf{g}^k \cdot \mathbf{b}_j \quad (17)$$

Here $\mathbf{G}_k = \frac{\partial \hat{\mathbf{R}}}{\partial x_k}$ is the covariant basis vector of the deformed configuration. One can obtain \mathbf{G}_k with the help of the definition of so-called generalized 2-D strains [11,6], given by

$$\mathbf{R}_{,\alpha} = A_\alpha (\mathbf{B}_\alpha + \varepsilon_{\alpha\beta} \mathbf{B}_\beta) \quad (18)$$

and

$$\mathbf{B}_{i,\alpha} = A_\alpha (-K_{\alpha 2} \mathbf{B}_1 + K_{\alpha 1} \mathbf{B}_2 + K_{\alpha 3} \mathbf{B}_3) \times \mathbf{B}_i \quad (19)$$

where $\varepsilon_{\alpha\beta}$ are the 2-D in-plane strains and of which the order is denoted by ε , and K_{ij} are the curvatures of the deformed surface which are the summation

of curvatures of undeformed geometry k_{ij} and curvatures introduced by the deformation κ_{ij} of which the order is denoted by ε/h . Both $\varepsilon_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ are termed as 2-D generalized strains. Here one is free to set $\varepsilon_{12} = \varepsilon_{21}$, *i.e.*

$$\frac{\mathbf{B}_1 \cdot \mathbf{R}_{,2}}{A_2} = \frac{\mathbf{B}_2 \cdot \mathbf{R}_{,1}}{A_1} \quad (20)$$

which can serve as another constraint to specify the global rotation of the triad \mathbf{B}_i .

With the assumption that the strain is small compared to unity, which has the effect of removing all the terms that are products of the warping and the generalized strains and with the help of Eqs. (12), (16), (17), (18) and (19), one can obtain the 3-D strain field as

$$\begin{aligned} \Gamma_{11} &= \frac{\varepsilon_{11} + x_3 \kappa_{11} + w_{1,1}/A_1 + w_3 k_{11} - w_2 k_{13}}{1 + x_3 k_{11}} \\ 2\Gamma_{12} &= \frac{\varepsilon_{21} + x_3 \kappa_{21} + w_{1,2}/A_2 - w_2 k_{23}}{1 + x_3 k_{22}} \\ &\quad + \frac{\varepsilon_{12} + x_3 \kappa_{12} + w_{2,1}/A_1 + w_1 k_{13}}{1 + x_3 k_{11}} \\ \Gamma_{22} &= \frac{\varepsilon_{22} + x_3 \kappa_{22} + w_{2,2}/A_2 + w_3 k_{22} + w_1 k_{23}}{1 + x_3 k_{22}} \\ 2\Gamma_{13} &= w_{1,3} + \frac{w_{3,1}/A_1 - w_1 k_{11}}{1 + x_3 k_{11}} \\ 2\Gamma_{23} &= w_{2,3} + \frac{w_{3,2}/A_2 - w_2 k_{22}}{1 + x_3 k_{22}} \\ \Gamma_{33} &= w_{3,3} \end{aligned} \quad (21)$$

Until now, the analysis is as general as a small-strain, geometrically nonlinear theory can be. However, to seek a constitutive model relating the generalized strains and stress resultants, it is inevitable that one has to make some approximations, as mentioned before. It is reasonable to keep the approximations in the constitutive relations since even the original 3-D constitutive model is unavoidably approximate.

For most engineering structures, ε is a small parameter in the order of $10^{-5} \sim 10^{-3}$ and it makes no sense to keep terms in the order of ε in comparison with unity. This fact has already been taken advantage of to derive Eq. (21). For thin shells, $h/R \sim 10^{-2}$, the components in the order of h/R can be discarded. For shells with moderate thickness ($h/R \sim 10^{-1}$), the correction of h/R (geometrical refinement) is necessary. As mentioned in Ref. [2], numerical examples show that 2-D shell models some times give satisfactory results down to $h/l \sim 0.5$. This fact convinces us to construct a 2-D theory taking account of h/l and $(h/l)^2$ corrections (shear refinement). However we do not include any

correction from $h^2/(Rl)$. This means that the present theory is asymptotically correct for $R > l^2/h$. And also as we are interested in the central solution with geometrical and shear refinements for regular shells, we assume that the initial curvatures k_{ij} and Lamé parameters A_α are slowly varying or constant. This assumption will result in neglecting all the derivatives of these quantities with respect to in-plane coordinates in the formulation. It is also been proved by the compatibility equations of 2-D strains that $\kappa_{21} - \kappa_{12}$ is of the order $1/R$ or $\varepsilon h/l^2$ which only makes a contribution of $(h/R)^2$ or $\varepsilon h^3/l^2/R$ and outside the range of our approximation [16]. It is worthwhile to emphasize that these assumptions are necessary only for the sake of seeking a Reissner-Mindlin like constitutive model. The approximations in determining the 2-D constitutive law are the only approximations in the 2-D theory.

Having made above approximations, one can express the 3-D strain field with keeping the terms of order h/R , h/l and $(h/l)^2$ in matrix form as:

$$\Gamma = \Gamma_h w + \Gamma_\epsilon \epsilon + \Gamma_{Rh} w + \Gamma_{R\epsilon} \epsilon + \Gamma_{l_1} w_{;1} + \Gamma_{l_2} w_{;2} \quad (22)$$

where

$$\begin{aligned} \Gamma &= [\Gamma_{11} \ 2\Gamma_{12} \ \Gamma_{22} \ 2\Gamma_{13} \ 2\Gamma_{23} \ \Gamma_{33}]^T \\ w &= [w_1 \ w_2 \ w_3]^T \\ \epsilon &= [\varepsilon_{11} \ 2\varepsilon_{12} \ \varepsilon_{22} \ \kappa_{11} \ \kappa_{12} + \kappa_{21} \ \kappa_{22}]^T \end{aligned} \quad (23)$$

and $(\)_{;\alpha} = \frac{\partial}{\partial x_\alpha} \frac{1}{A_\alpha}$. Since A_α are constants all the rules of differentiation are applicable to the newly-introduced notation. All the operators are defined as:

$$\Gamma_h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} \end{bmatrix} \quad \Gamma_{l_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Gamma_{l_2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (24)$$

$$\Gamma_\epsilon = \begin{bmatrix} 1 & 0 & 0 & x_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & x_3 & 0 \\ 0 & 0 & 1 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \Gamma_{Rh} = \begin{bmatrix} 0 & 0 & k_{11} \\ 0 & 0 & 0 \\ 0 & 0 & k_{22} \\ -k_{11} & 0 & 0 \\ 0 & -k_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (25)$$

$$\Gamma_{R\epsilon} = -x_3 \begin{bmatrix} k_{11} & 0 & 0 & x_3 k_{11} & 0 & 0 \\ 0 & \frac{k_{11}+k_{22}}{2} & 0 & 0 & x_3 \frac{k_{11}+k_{22}}{2} & 0 \\ 0 & 0 & k_{22} & 0 & 0 & x_3 k_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (26)$$

Now, the total strain energy of the shell structure can be expressed as:

$$J = \frac{1}{2} \int_v \Gamma^T D \Gamma \mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 dx_1 dx_2 dx_3 = \frac{1}{2} \int_s \langle \Gamma^T D \Gamma \rho \rangle ds \quad (27)$$

where v is the volume occupied by the 3-D body in the undeformed configuration, s is the surface stretched by the undeformed reference surface and

$$\rho = \frac{\mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3}{|\mathbf{a}_1 \times \mathbf{a}_2|} = 1 + x_3(k_{11} + k_{22}) + O\left(\frac{h^2}{R^2}\right) \quad (28)$$

The strain energy per unit area (which is the same as the strain energy for the deformation of the normal-line element) is

$$U = \frac{1}{2} \langle \Gamma^T D \Gamma \rho \rangle \quad (29)$$

where D is the 3-D 6×6 material matrix, which comes from the fourth order elasticity tensor expressed in the basis \mathbf{b}_i . This matrix is in general fully populated. However, if it is desired to model laminated composite shells in which each lamina exhibits a monoclinic symmetry about its own mid-surface (for which the material matrix is determined by 13 constants instead of 21) and is rotated about the local normal to be a layer in the composite laminated shell, some part of this material matrix will always vanish no matter what the layup angle is [18].

To deal with applied loads, we will at first leave open the existence of a potential energy and develop instead the virtual work of the applied loads. The virtual displacement is taken as the Lagrangean variation of the displacement field, such that

$$\delta \hat{\mathbf{R}} = \overline{\delta q}_{B_i} \mathbf{B}_i + x_3 \overline{\delta \psi}_{B_i} \mathbf{B}_i \times \mathbf{B}_3 + \delta w_i \mathbf{B}_i + \overline{\delta \psi}_{B_i} \mathbf{B}_i \times w_j \mathbf{B}_j \quad (30)$$

where the virtual displacement of the reference surface is given by

$$\overline{\delta q}_{B_i} = \delta \mathbf{u} \cdot \mathbf{B}_i \quad (31)$$

and the virtual rotation of the reference surface is defined such that

$$\delta \mathbf{B}_i = (-\overline{\delta\psi}_{B\beta} \mathbf{B}_\beta \times \mathbf{B}_3 + \overline{\delta\psi}_{B3} \mathbf{B}_3) \times \mathbf{B}_i \quad (32)$$

Since the strain is small, one may safely ignore products of the warping and the loading in the virtual rotation term. Then, the work done through a virtual displacement due to the applied loads $\tau_i \mathbf{B}_i$ at the top surface and $\beta_i \mathbf{B}_i$ at the bottom surface and body force $\phi_i \mathbf{B}_i$ through the thickness is

$$\overline{\delta W} = (\tau_i + \beta_i + \langle \phi_i \rangle) \overline{\delta q}_{B_i} + \overline{\delta\psi}_{B\alpha} \left[\frac{h}{2} (\tau_\alpha - \beta_\alpha) + \langle x_3 \phi_\alpha \rangle \right] + \delta (\tau_i w_i^+ + \beta_i w_i^- + \langle \phi_i w_i \rangle) \quad (33)$$

where τ_i , β_i , and ϕ_i are taken to be independent of the deformation, $(\)^+ = (\)|_{x_3=\frac{h}{2}}$, and $(\)^- = (\)|_{x_3=-\frac{h}{2}}$. By introducing column matrices $\overline{\delta q}$, $\overline{\delta\psi}$, τ , β , and ϕ , which are formed by stacking the three elements associated with indexed symbols of the same names, and using Eqs. (1), (13), and (14) one may write the virtual work in a matrix form, so that

$$\overline{\delta W} = \overline{\delta q}^T f + \overline{\delta\psi}^T m + \delta (\tau^T w^+ + \beta^T w^- + \langle \phi^T w \rangle) \quad (34)$$

where

$$f = \tau + \beta + \langle \phi \rangle$$

$$m = \begin{Bmatrix} \frac{h}{2} (\tau_1 - \beta_1) + \langle x_3 \phi_1 \rangle \\ \frac{h}{2} (\tau_2 - \beta_2) + \langle x_3 \phi_2 \rangle \\ 0 \end{Bmatrix} \quad (35)$$

Here because loading is of the order h/l or higher, we do not include any geometrical correction h/R into the virtual work which is acceptable in the accuracy of our approximation. The complete statement of the problem can now be presented in terms of the principle of virtual work, such that

$$\delta U - \overline{\delta W} = 0 \quad (36)$$

We have three kinds of virtual quantities here: the virtual displacement $\overline{\delta q}$, the virtual rotation $\overline{\delta\psi}$ and the variation of warping field δw_i . The first two quantities will be handled by 2-D shell theory. And δw_i is the only unknown quantity to be determined in the process of modeling. It can be observed that in spite of the possibility of accounting for nonconservative forces in the above, the problem that governs the warping is conservative. Thus, one can pose the problem that governs the warping as the minimization of a total potential functional

$$\Pi = U + W \quad (37)$$

so that

$$\delta \Pi = 0 \quad (38)$$

in which only the warping displacement is varied, subject to the constraints Eq. (15). This implies that

$$W = -\tau^T w^+ - \beta^T w^- - \langle \phi^T w \rangle \quad (39)$$

Below, for simplicity of terminology, we will refer to Π as the total potential energy, or the total energy.

By the principle of minimum total potential energy, one can solve for the unknown warping functions by minimizing the functional in Eq. (37) subject to the constraints of Eq. (15). However, this problem is still 3-D. If we attempt to solve this problem directly, we will meet the same difficulty as solving any full 3-D elasticity problem. Fortunately, as shown in Ref. [18], the VAM can be used to calculate the 3-D warping functions asymptotically. To deal with numerous layers and to be compatible with 2-D FEM solvers, a finite element discretization is used to solve the minimization problem. A five-noded isoparametric element is recommended because we will need the warping functions up to the second-order approximation which are piecewise polynomials of the fourth order. Discretizing the transverse normal line into 1-D finite elements, one can express the warping field as

$$w(x_i) = S(x_3)V(x_1, x_2) \quad (40)$$

where S is the shape function and V is the nodal value of warping field along the transverse normal. Now one is ready to use VAM to solve the unknown warping field asymptotically.

3 Dimensional Reduction

Now, to rigorously reduce the original 3-D problem to a 2-D shell problem, one must attempt to reproduce the energy stored in the 3-D structure in a 2-D formulation. This dimensional reduction can only be done approximately, and one way to do it is by taking advantage of the smallness of h/l and h/R . Another small parameter is the order of the maximum strain in the shell, which we denote here as ε . The small parameter ε has already been taken advantage of when we derived Eq. (22). The approximate energy we are interested in is

$$\Pi = \mu\varepsilon^2 \left[O(1) + O\left(\frac{h}{R}\right) + O\left(\frac{h}{l}\right) + O\left(\frac{h^2}{l^2}\right) \right] \quad (41)$$

where μ represents the material elastic constants, all of which are assumed to be of the same order. All terms of higher order are discarded, and the terms retained are sufficient for the purpose of creating a shallow shell model of the Reissner-Mindlin type. Substituting Eq. (40) into Eq. (37), one can express

the total energy within the accuracy of our approximation in discretized form as

$$\begin{aligned}
2\Pi = & V^T E V + 2V^T (D_{h\epsilon}\epsilon + D_{Rh\epsilon}\epsilon \\
& + D_{hRh}V + D_{hR\epsilon}\epsilon + D_{hl_1}V_{;1} + D_{hl_2}V_{;2}) \\
& + \epsilon^T (D_{\epsilon\epsilon} + 2D_{\epsilon R\epsilon})\epsilon + V_{;1}^T D_{l_1 l_1} V_{;1} + V_{;2}^T D_{l_2 l_2} V_{;2} \\
& + 2(V_{;1}^T D_{l_1 \epsilon}\epsilon + V_{;2}^T D_{l_2 \epsilon}\epsilon + V_{;1}^T D_{l_1 l_2} V_{;2}) + 2V^T L
\end{aligned} \tag{42}$$

where L contains the load related terms such that

$$L = -S^{+T}\tau - S^{-T}\beta - \langle S^T \phi \rangle \tag{43}$$

The new matrices carry the properties of both the geometry and material:

$$\begin{aligned}
E &= \langle [\Gamma_h S]^T D[\Gamma_h S] \rho \rangle & D_{h\epsilon} &= \langle [\Gamma_h S]^T D\Gamma_\epsilon \rho \rangle \\
D_{hl_1} &= \langle [\Gamma_h S]^T D[\Gamma_{l_1} S] \rangle & D_{hl_2} &= \langle [\Gamma_h S]^T D[\Gamma_{l_2} S] \rangle \\
D_{\epsilon\epsilon} &= \langle \Gamma_\epsilon^T D\Gamma_\epsilon \rho \rangle & D_{l_1 l_1} &= \langle [\Gamma_{l_1} S]^T D[\Gamma_{l_1} S] \rangle \\
D_{l_1 l_2} &= \langle [\Gamma_{l_1} S]^T D[\Gamma_{l_2} S] \rangle & D_{l_2 l_2} &= \langle [\Gamma_{l_2} S]^T D[\Gamma_{l_2} S] \rangle \\
D_{hRh} &= \langle [\Gamma_h S]^T D[\Gamma_{Rh} S] \rangle & D_{hR\epsilon} &= \langle [\Gamma_h S]^T D\Gamma_{R\epsilon} \rangle \\
D_{Rh\epsilon} &= \langle [\Gamma_{Rh} S]^T D\Gamma_\epsilon \rangle & D_{\epsilon R\epsilon} &= \langle \Gamma_\epsilon^T D\Gamma_{R\epsilon} \rangle
\end{aligned} \tag{44}$$

One can observe that the first term of Eq. (22) has order $\frac{\|V\|}{h}$ and the last two terms have order $\frac{\|V\|}{l}$ which is clearly one order of h/l higher than the first term. This observation allows us to avoid dealing with derivatives of unknown functions with respect to in-plane coordinates. As mentioned before, although reduced-order models based on *ad hoc* kinematic assumptions regularly appear in the literature, there is no basis whatsoever to justify such assumptions. Rather, in this work, the VAM will be used to mathematically perform a dimensional reduction of the 3-D problem to a series of 2-D models. One can refer to Ref. [1] for a brief introduction of the VAM. To proceed by this method, one has to assess and keep track of the order of all the quantities in the formulation. Following Ref. [14], the quantities of interest have the following orders:

$$\begin{aligned}
\epsilon_{\alpha\beta} &\sim h\kappa_{\alpha\beta} \sim \varepsilon & f_3 &\sim \mu(h/l)^2\varepsilon \\
f_\alpha &\sim \mu(h/l)\varepsilon & m_\alpha &\sim \mu h(h/l)\varepsilon
\end{aligned} \tag{45}$$

It is noted that $m_3 = 0$ if the elastic body is not subject to body moments.

The VAM requires one to find the leading terms of the functional according to the different orders. The total potential energy consists of quadratic expressions involving the warping and the generalized strains. In addition there are terms that involve the loading along with interaction terms between the warping and the both of the other types of quantities. For the zeroth-order

approximation, these leading terms of Eq. (42) are

$$2\Pi_0 = V^T E_0 V + 2V^T D_{h\epsilon 0} \epsilon + \epsilon^T D_{\epsilon\epsilon 0} \epsilon \quad (46)$$

where E_0 , $D_{h\epsilon 0}$ and $D_{\epsilon\epsilon 0}$ are the E , $D_{h\epsilon}$, and $D_{\epsilon\epsilon}$ matrices defined in Eq. (44) with $\rho = 1$ (no geometrical correction). The discretized form of Eq. (15) can be written as

$$V^T H \psi = 0 \quad (47)$$

where $H = \langle S^T S \rangle$ and ψ is the normalized kernel matrix of E_0 such that $\psi^T H \psi = I$. Our problem has now been transformed into the numerical minimization of Eq. (46) subject to constraints Eq. (47). The Euler-Lagrange equation for this problem can be obtained by usual procedure of the calculus of variations with the aid of a Lagrange multiplier as follows:

$$E_0 V + D_{h\epsilon 0} \epsilon = H \psi \Lambda \quad (48)$$

Considering the properties of the kernel matrix ψ , one calculates the Lagrange multiplier Λ as

$$\Lambda = \psi^T D_{h\epsilon 0} \epsilon \quad (49)$$

Substituting Eq. (49) back into Eq. (48), we obtain

$$E_0 V = (H \psi \psi^T - I) D_{h\epsilon 0} \epsilon \quad (50)$$

There exists a unique solution linearly independent of the null space of E_0 for V because the right-hand-side of Eq. (50) is orthogonal to the null space. Since the solution is unique, we can choose any convenient constraints to make the problem determined. In our implementation, we arbitrarily constrain three degrees of freedom to obtain a solution V^* for the linear system, so that the final solution can be written as

$$V = V^* + \psi \lambda \quad (51)$$

where λ can be determined by Eq. (47) as

$$\lambda = -\psi^T H V^* \quad (52)$$

Hence the final solution minimizing the functional Eq. (46) subject to constraints Eq. (47) is

$$V = (I - \psi \psi^T H) V^* = \hat{V}_0 \epsilon = V_0 \quad (53)$$

Substituting Eq. (53) back into Eq. (42), one can obtain the total energy asymptotically correct through the order of $\mu \epsilon^2$ as

$$2\Pi_0 = \epsilon^T (\hat{V}_0^T D_{h\epsilon 0} + D_{\epsilon\epsilon 0}) \epsilon = \epsilon^T A \epsilon \quad (54)$$

As pointed out in Ref. [18], the energy of this approximation coincides with classical laminated shell theories. However, we do not use *ad hoc* kinematic

assumptions such as the Kirchhoff assumption to obtain this result. Although the energy is the same, the transverse normal strain from our zeroth-order approximation is not zero. The result is also the same as the zeroth-order approximation of plate theory because we have not yet included the geometrical correction due to the initial curvatures.

Although shell theory based on this zeroth-order approximation can do a good job in predicting the global deformation and in-plane quantities for thin structures, refined theories taking advantage of small parameters h/R and h/l are required for moderately thick shells to give a better prediction of global deformation, in-plane quantities and especially out-of-plane stresses and strains (σ_{i3} , Γ_{i3}). Let us obtain the correction coming from h/R first to include the effect of initial curvatures of the structure. Usually to find the refinement, one needs to calculate the refined warping functions based on the next approximation. However, since we are only interested in obtaining an energy asymptotically correct up to the order of h/R which is sufficient for most of the engineering applications, it is unnecessary to calculate the refined warping with respect to h/R which makes no contribution to the energy up to the order of h/R . At this stage we will postpone the consideration of the load contribution to the step of h/l correction. The strain energy asymptotically correct up to the order of h/R can be expressed as

$$\begin{aligned} 2\Pi_R = & \epsilon^T (\hat{V}_0^T D_{he0} + D_{ee0}) \epsilon + \epsilon^T (\hat{V}_0^T E^* \hat{V}_0 + D_{ee}^*) \epsilon \\ & + 2\epsilon^T (\hat{V}_0^T D_{hRe} + \hat{V}_0^T D_{hRh} \hat{V}_0 + D_{eRe} + \hat{V}_0^T D_{Rhe} + \hat{V}_0^T D_{he}^*) \epsilon \end{aligned} \quad (55)$$

where

$$\begin{aligned} E^* &= \langle [\Gamma_h S]^T D [\Gamma_h S] (\rho - 1) \rangle \\ D_{he}^* &= \langle [\Gamma_h S]^T D \Gamma_\epsilon (\rho - 1) \rangle \\ D_{ee}^* &= \langle \Gamma_\epsilon^T D \Gamma_\epsilon (\rho - 1) \rangle \end{aligned} \quad (56)$$

To account for transverse shear deformation, one needs to find an energy that is asymptotically correct through the order of $(h/l)^2$ relative to the leading terms. To obtain this refinement, one needs to find the refined warping field in the order of h/l . Perturbing the zeroth-order result with a warping field V_1 which is of the order of $(h/l)V_0$, one obtains:

$$V = V_0 + V_1 \quad (57)$$

Substituting Eq. (57) back into Eq. (42), one can obtain the leading terms for the first-order approximation as

$$2\Pi_1^* = V_1^T E V_1 + 2V_1^T D_{1\epsilon;1} + 2V_1^T D_{2\epsilon;2} + 2V_1^T L \quad (58)$$

where

$$D_1 = (D_{hl_1} - D_{hl_1}^T) \hat{V}_0 - D_{l_1 \epsilon} \quad (59)$$

$$D_2 = (D_{hl_2} - D_{hl_2}^T) \hat{V}_0 - D_{l_2 \epsilon} \quad (60)$$

It is understood that the order of the loads in Eq. (45) is associated with warping functions of different orders, as shown in Ref. [18]. For example, L in Eq. (58) is only comprised of the in-plane components of the applied loads. Integration by parts with respect to in-plane coordinates is used here and hereafter whenever it is convenient for the derivation, because the present goal is to obtain an interior solution for the shell without consideration of edge effects.

Similarly as in the zeroth-order approximation, one can solve the first-order warping field as

$$V_1 = V_{11} \epsilon_{;1} + V_{12} \epsilon_{;2} + V_{1L} \quad (61)$$

and obtain a total energy that is asymptotically correct up to the order of $\mu(h/l)^2 \epsilon$ and $\mu h/R \epsilon$, given by

$$2\Pi_1 = \epsilon^T A_R \epsilon + \epsilon_{;1}^T B \epsilon_{;1} + 2\epsilon_{;1}^T C \epsilon_{;2} + \epsilon_{;2}^T \mathcal{D} \epsilon_{;2} + 2\epsilon^T F + P \quad (62)$$

where

$$\begin{aligned} A_R &= A + \hat{V}_0^T E^* \hat{V}_0 + D_{\epsilon\epsilon}^* + 2(\hat{V}_0^T D_{hR\epsilon} + \hat{V}_0^T D_{hRh} \hat{V}_0 + D_{\epsilon R\epsilon} + \hat{V}_0^T D_{Rh\epsilon} + \hat{V}_0^T D_{h\epsilon}^*) \\ B &= \hat{V}_0^T D_{l_1 l_1} \hat{V}_0 + V_{11}^T D_1 \\ C &= \hat{V}_0^T D_{l_1 l_2} \hat{V}_0 + \frac{1}{2}(V_{11}^T D_2 + D_1^T V_{12}) \\ \mathcal{D} &= \hat{V}_0^T D_{l_2 l_2} \hat{V}_0 + V_{12}^T D_2 \\ F &= \hat{V}_0^T L - \frac{1}{2}(D_1^T V_{1L;1} + V_{11}^T L_{;1} + D_2^T V_{1L;2} + V_{12}^T L_{;2}) \\ P &= V_{1L}^T L \end{aligned} \quad (63)$$

Here we have already taken advantage of monoclinic symmetry to obtain the asymptotically correct energy in Eq. (62). It is noted that P is a set of quadratic terms from the applied loads and it cannot be varied in the 2-D model. When there is no load, this term will vanish. It comes from the applied load and the warping of refined approximations introduced by the applied load. Also, the applied loads should not vary rapidly over the shell reference surface. Otherwise, the derivative terms contained in F will not be higher-order terms to meet the requirement of asymptotical correctness.

4 Transforming Into a Reissner-Mindlin Model

Although Eq. (62) is asymptotically correct through the second order and straightforward use of this strain energy expression is possible as mentioned in Ref. [14], it involves more complicated boundary conditions than necessary since it contains derivatives of the generalized strain measures. To obtain an energy functional that is of practical use, one can transform the present approximation into a Reissner-Mindlin model, which is the most commonly used model in practice.

In a Reissner-Mindlin model, there are two additional degrees of freedom, which are the transverse shear strains. These are incorporated into the rotation of a local line element through the thickness. If we introduce another triad \mathbf{B}_i^* for the deformed Reissner-Mindlin shell, the definition of 2-D strains becomes

$$\mathbf{R}_{,\alpha} = A_\alpha(\mathbf{B}_\alpha^* + \varepsilon_{\alpha\beta}^* \mathbf{B}_\beta^* + 2\gamma_{\alpha 3} \mathbf{B}_3^*) \quad (64)$$

$$\mathbf{B}_{i,\alpha}^* = A_\alpha(-K_{\alpha 2}^* \mathbf{B}_1^* + K_{\alpha 1}^* \mathbf{B}_2^* + K_{\alpha 3}^* \mathbf{B}_3^*) \times \mathbf{B}_i^* \quad (65)$$

where the transverse shear strains are $\gamma = [2\gamma_{13} \ 2\gamma_{23}]^T$. Since \mathbf{B}_i^* is uniquely determined by \mathbf{B}_i and γ , one can derive the following kinematic identity between the strains measures \mathcal{R} of Reissner-Mindlin shell and ϵ

$$\epsilon = \mathcal{R} - \mathcal{D}_1 \gamma_{;1} - \mathcal{D}_2 \gamma_{;2} \quad (66)$$

where

$$\begin{aligned} \mathcal{D}_1 &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T \\ \mathcal{D}_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \\ \mathcal{R} &= [\varepsilon_{11}^* \ 2\varepsilon_{12}^* \ \varepsilon_{22}^* \ \kappa_{11}^* \ \kappa_{12}^* + \kappa_{21}^* \ \kappa_{22}^*]^T \end{aligned} \quad (67)$$

Now one can rewrite the strain energy expressed in Eq. (62), correct to the orders of interest according to Eq. (41), in terms of strains of the Reissner-Mindlin model as

$$\begin{aligned} 2\Pi_1 &= \mathcal{R}^T A_R \mathcal{R} - 2\mathcal{R}^T A \mathcal{D}_1 \gamma_{;1} - 2\mathcal{R}^T A \mathcal{D}_2 \gamma_{;2} \\ &\quad + \mathcal{R}_{;1}^T B \mathcal{R}_{;1} + 2\mathcal{R}_{;1}^T C \mathcal{R}_{;2} + \mathcal{R}_{;2}^T D \mathcal{R}_{;2} + 2\mathcal{R}^T F + P \end{aligned} \quad (68)$$

The generalized Reissner-Mindlin model of practical use can be of the form

$$2\Pi_{\mathcal{R}} = \mathcal{R}^T A_R \mathcal{R} + \gamma^T G \gamma + 2\mathcal{R}^T F_{\mathcal{R}} + 2\gamma^T F_\gamma \quad (69)$$

To find an equivalent Reissner-Mindlin model, Eq. (69), for Eq. (68), one has to eliminate all partial derivatives of the classical 2-D strain measures $\mathcal{R}_{;\alpha}$. The equilibrium equations are used to achieve this purpose. From the two equilibrium equations balancing bending moments with applied moments m_α which is calculated from Eq. (35), one can obtain the following formula

$$G\gamma + F_\gamma = \mathcal{D}_1^T A \mathcal{R}_{;1} + \mathcal{D}_2^T A \mathcal{R}_{;2} + \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} \quad (70)$$

Using Eq. (70), one can rewrite Eq. (68) as

$$2U = \mathcal{R}^T A \mathcal{R} + \gamma^T G \gamma + 2\mathcal{R}^T F + \bar{P} + U^* \quad (71)$$

where

$$U^* = \mathcal{R}_{;1}^T \bar{B} \mathcal{R}_{;1} + 2\mathcal{R}_{;1}^T \bar{C} \mathcal{R}_{;2} + \mathcal{R}_{;2}^T \bar{D} \mathcal{R}_{;2} \quad (72)$$

and

$$\begin{aligned} \bar{B} &= B + A \mathcal{D}_1 G^{-1} \mathcal{D}_1^T A \\ \bar{C} &= C + A \mathcal{D}_1 G^{-1} \mathcal{D}_2^T A \\ \bar{D} &= D + A \mathcal{D}_2 G^{-1} \mathcal{D}_2^T A \\ \bar{P} &= P - \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix}^T G^{-1} \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} \end{aligned} \quad (73)$$

If we can drive U^* to be zero for any \mathcal{R} , then we have found an asymptotically correct Reissner-Mindlin shell model. For general anisotropic shells, this term will not be zero; but we can minimize the error to obtain a Reissner-Mindlin model that is as close to asymptotical correctness as possible. The accuracy of the Reissner-Mindlin model depends on how close to zero one can drive this term of the energy.

One could proceed with the optimization at this point, but the problem will lead to a least squares solution for 3 unknowns (the shear stiffness matrix G) from a linear system of 78 equations (12×12 and symmetric). This optimization problem is far too rigid to be practical. A better solution will be obtained if we can bring more unknowns into the problem. As stated in Ref. [15], there is no unique shell theory of a given order. One can relax the constraints in Eq. (15) to be $\langle w_i \rangle = \text{const}$ and still obtain an asymptotically correct strain energy. Since the zeroth-order approximation gives us an asymptotic model corresponding to classical shell theory, we only relax the constraints for the first-order approximation. This relaxation will modify the warping field to be

$$\bar{V}_1 = V_{11}\epsilon_{;1} + V_{12}\epsilon_{;2} + V_{1L} + L_1\epsilon_{;1} + L_2\epsilon_{;2} \quad (74)$$

where L_1, L_2 consist of 24 constants. The remaining energy U^* will also be modified to be

$$U^* = \mathcal{R}_{;1}^T \hat{B} \mathcal{R}_{;1} + 2\mathcal{R}_{;1}^T \hat{C} \mathcal{R}_{;2} + \mathcal{R}_{;2}^T \hat{D} \mathcal{R}_{;2} \quad (75)$$

and

$$\begin{aligned} \hat{B} &= \bar{B} + 2L_1^T D_1 \\ \hat{C} &= \bar{C} + (L_1^T D_2 + D_1^T L_2) \\ \hat{D} &= \bar{D} + 2L_2^T D_2 \end{aligned} \quad (76)$$

Since now we have 27 unknowns, the optimization is much more flexible. It can give us a more optimal solution for the shear stiffness matrix G to fit the second-order, asymptotically-correct energy into a Reissner-Mindlin model. In other words, here we have found the Reissner-Mindlin model that describes as closely as possible the 2-D energy that is asymptotically correct through the second order in h/l . However, the asymptotical correctness of the warping field can only be ascertained after obtaining another higher-order approximation, which will be discussed in the next section.

And after minimizing U^* , the Reissner-Mindlin model to be used for 2-D shell problem can be expressed as:

$$2\Pi_{\mathcal{R}} = \mathcal{R}^T A_R \mathcal{R} + \gamma^T G \gamma + 2\mathcal{R}^T F \quad (77)$$

where from Eq. (71) the quadratic loads term of \bar{P} is dropped because it will not affect the 2-D governing equations. It should be noted that the load-related terms in F are a new feature in the present development. One must modify traditional Reissner-Mindlin shell solvers to accommodate these terms. This modification is not difficult and has a form similar to terms that must be included when considering thermal effects or actuated materials.

5 Recovery Relations

From the above, we have obtained a Reissner-Mindlin shell model which is as close as possible to being asymptotically correct in the sense of matching the total potential energy. The stiffness matrices obtained, A_R and G , along with the load related term F can be used as input for a shell theory derived from the total energy as shown in Eq. (77). A geometrically nonlinear shell theory that is consistent with present formulation has been developed in [16].

In many applications, however, while it is necessary to accurately calculate the 2-D displacement field of composite shells, this is not sufficient. Ultimately, the fidelity of a reduced-order model such as this depends on how well it can predict the 3-D results in the original 3-D structure. Hence recovery relations should be provided to complete the reduced-order model and the results then

compared with those of the original 3-D model. By recovery relations, then, we mean closed-form expressions for 3-D displacement, strain, and stress fields in terms of 2-D quantities and x_3 .

For a strain energy that is asymptotically correct through the second order, we can recover the 3-D displacement, strain and stress fields only through the first order in a strict sense of asymptotical correctness. Using Eqs. (1), (13), and (14), one can recover the 3-D displacement field through the first order as

$$U_{3d} = u_{2d} + x_3 \begin{bmatrix} C_{31} \\ C_{32} \\ C_{33} - 1 \end{bmatrix} + SV_0 + S\bar{V}_1 \quad (78)$$

where U_{3d} is the column matrix of 3-D displacements and u_{2d} is the shell displacements. C_{ij} are the components of global rotation tensor from Eq. (13). And from Eq. (22), one can recover the 3-D strain field through the first order as

$$\Gamma = \Gamma_h S(V_0 + \bar{V}_1) + \Gamma_\epsilon \epsilon + \Gamma_{l_1} SV_{0;1} + \Gamma_{l_2} SV_{0;2} + \Gamma_{Rh} V_0 + \Gamma_{R\epsilon} \epsilon \quad (79)$$

Then, one can use the 3-D constitutive law to obtain 3-D stresses σ_{ij} .

Since we have obtained an optimum shear stiffness matrix G , some of the recovered 3-D results through the first order are indeed better than classical theory and conventional first-order shear deformation theory. However, the transverse normal components of strain and stress (*i.e.* Γ_{33} and σ_{33}) do not agree well at all. Let us recall, that the Reissner-Mindlin theory that has been constructed only ensures a good fit with the asymptotically correct 3-D displacement field of the first order (while energy is approximated to the second order). Thus, in order to obtain recovery relations that are valid to the same order as the energy, the VAM iteration needs to be applied one more time.

Using the same procedure listed in previous section, the second-order warping can be obtained and expressed symbolically as

$$V_2 = V_{21}\epsilon_{;11} + V_{22}\epsilon_{;12} + V_{23}\epsilon_{;22} \quad (80)$$

Eq. (80) is obtained by taking the original first-order warping V_1 to be the result of the first-order approximation. It is clear that V_2 is one order higher than V_1 which confirms that V_1 is the first-order approximation. One might be tempted to use V_1 in the recovery relations. However, the VAM has split the original 3-D problem into two sets of problems. As far as recovery relations concerned, it is observed that the normal-line analysis can at best give us an approximate shape of the distribution of 3-D results. The 2-D shell analysis

will govern the global behavior of the structure. Since \bar{V}_1 is the warping that yields a Reissner-Mindlin shell model that is as close as possible to being asymptotically correct, we should still use \bar{V}_1 in the recovery relations instead of V_1 . By doing this, we choose to be more consistent with Reissner-Mindlin shell model while compromising somewhat on the asymptotical correctness of the shape of the distribution. It has been verified by numerical examples that this is a good choice.

Hence, we write the 3-D recovery relations for displacement through the second order as

$$U_{3d} = u_{2d} + x_3 \begin{bmatrix} C_{31} \\ C_{32} \\ C_{33} - 1 \end{bmatrix} + S(V_0 + \bar{V}_1 + V_2) \quad (81)$$

and the strain field through the second order is

$$\begin{aligned} \Gamma = & \Gamma_h S(V_0 + \bar{V}_1 + V_2) + \Gamma_\epsilon \epsilon + \Gamma_{l_1} S(V_{0;1} + \bar{V}_{1;1}) \\ & + \Gamma_{l_2} S(V_{0;2} + \bar{V}_{1;2}) + \Gamma_{Rh} V_0 + \Gamma_{R\epsilon} \epsilon \end{aligned} \quad (82)$$

Again the stresses through the second order can be obtained from use of the 3-D material law. It will be shown in the next section that the recovered 3-D results through the second order agree with the exact solutions very well.

6 Numerical Examples

The computer program, Variational Asymptotic shell and Shell Analysis (VAPAS), has been extended to include the present theory. Several numerical examples are given here to validate the theory and code. We investigate several cases for cylindrical bending of composite cylindrical shells. All the distributions of 3-D transverse shear and normal stresses through the thickness are compared with results from Classical Laminated Shell Theory (CLST), First-Order Shear Deformation Theory (FOSDT), and 3-D exact solutions based on Refs. [12,7]. The excellent agreement with exact solutions demonstrate that our Reissner-Mindlin shell model can be used to analyze composite shells efficiently and accurately.

The structure analyzed is a cylindrical shell (see Fig. 2) with $x_{1max} = R\phi$ along x_1 (the ‘‘lateral’’ direction) and infinite length in the x_2 direction (the ‘‘longitudinal’’ direction). The thickness of the shell is 1 in., and the radius of

the cylinder is R . The material properties of the laminated shell are

$$\begin{aligned} E_L &= 25 \times 10^6 \text{ psi} & E_T &= 10^6 \text{ psi} \\ G_{LT} &= 0.5 \times 10^6 \text{ psi} & G_{TT} &= 0.2 \times 10^6 \text{ psi} \\ \nu_{LT} &= \nu_{TT} = 0.25 \end{aligned}$$

The shell is simply supported and subjected to a sinusoidal surface loading of the form

$$\tau_3 = \beta_3 = \frac{p_0}{2} \sin\left(\frac{n\pi x_1}{R\phi}\right) \quad (83)$$

with $\tau_\alpha = \beta_\alpha = 0$.

Four different cases are investigated:

- case 1: nearly cross ply, $[89.5^\circ / -0.5^\circ]$ and $R = 4, n = 1, \phi = \pi/3$
- case 2: nearly cross ply, $[89.5^\circ / -0.5^\circ]$ and $R = 10, n = 2, \phi = \pi/5$
- case 3: symmetric nearly cross ply, $[0.5^\circ / 90.5^\circ / 90.5^\circ / 0.5^\circ]$ and $R = 10, n = 2, \phi = \pi/5$
- case 4: symmetric angle ply, $[30^\circ / -30^\circ / -30^\circ / 30^\circ]$ and $R = 10, n = 2, \phi = \pi/5$

It is noted that the reason we have changed the ply angle a small amount from the cross-ply case is because this will allow us to use a Mathematica™ code we developed based on Ref. [7] which does not apply to cross-ply case.

Although the Reissner-Mindlin model obtained above, Eq. (71), can be used along with the nonlinear 2-D shell theory derived previously, only geometrically linear examples are considered here for the purpose of comparing the results with the 3-D elasticity solution [12], [7] which is based on 3-D linear elasticity.

Transverse stress components σ_{i3} are very important for analyzing composite shells and common shell models fail to provide a good prediction for these quantities, especially, the transverse normal stress. Out-of-plane quantities are presented here to validate our shell model. Note that, because σ_{33} is a sine function of x_1 and $\sigma_{\alpha 3}$ are cosine functions of x_1 , their distributions through the thickness are picked up at the positions of x_1 where those stress components have the maximum values. The present results (dots in the plots), are compared with results from CLST (dash-dotted line), FOSDT (dashed line) and the exact solutions (solid line) generated here by Mathematica™. The results presented here are normalized as $\bar{\sigma}_{ij} = \frac{\sigma_{ij}}{p_0}$ and $\bar{z} = \frac{x_3}{h}$.

First, to compare with a very complex higher-order shear deformation theory (an “improved shear deformation theory” based on layer-wise theory enforcing transverse stress continuity) presented in Ref. [19], we study case 1. The out-of-plane stress components are shown in Figs. 3, 4 and 5. One can observe that σ_{13}

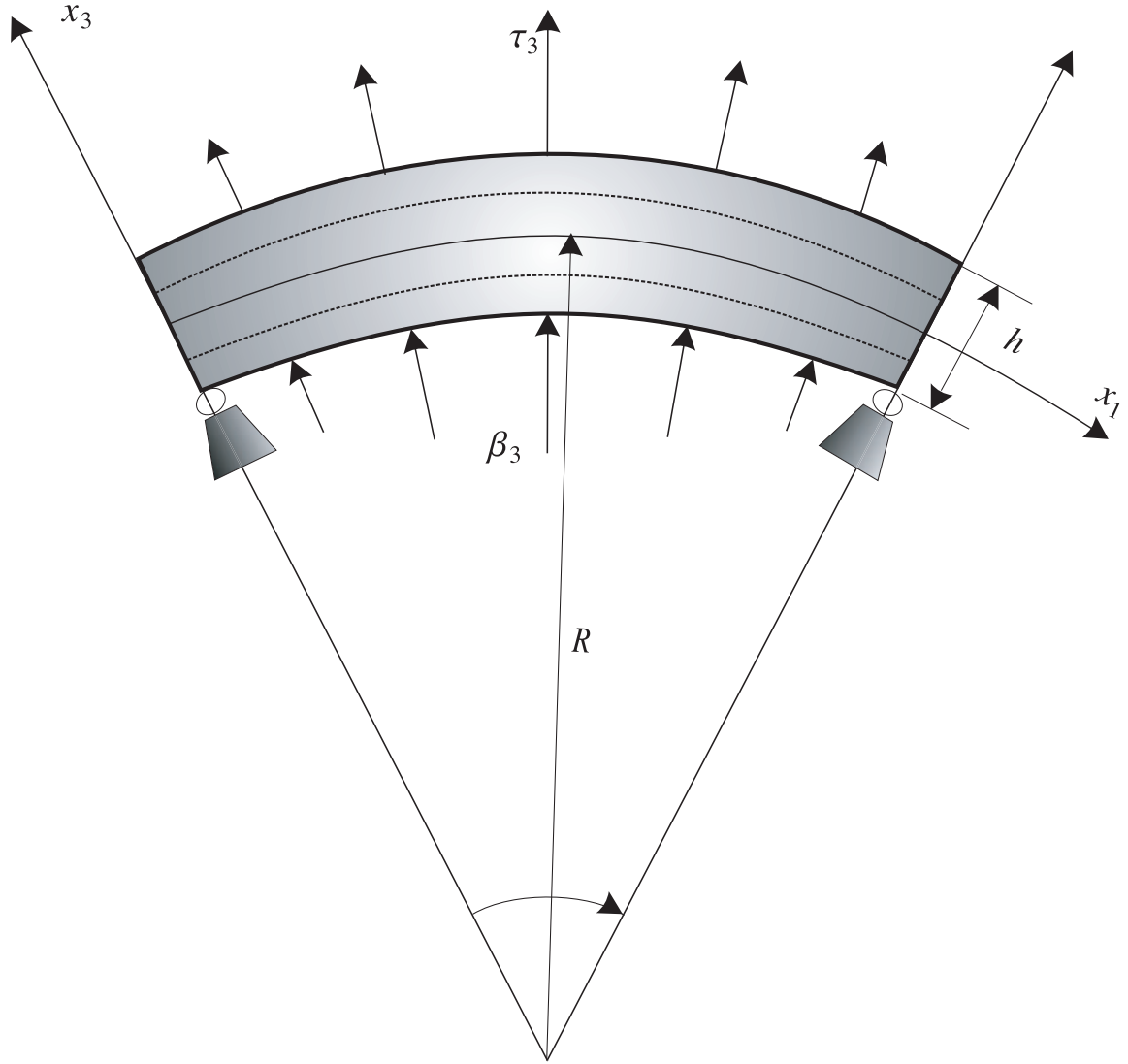


Fig. 2. Cylindrical bending of composite shell

and σ_{23} have a pretty good agreement with the exact solution. Considering the fact the our shell solver is still a simple single-layer Reissner-Mindlin model, the excellent agreement between the exact solution and the present theory is quite significant because in Ref. [19], the prediction of σ_{13} is off by almost 40% from the exact solution at the middle point of the upper layer.

However, the predicted transverse normal stress is not as accurate as the other components. The reason is that in this test case h/R is larger than h/L , where L is the wavelength of the loading function. Thus, this case does not satisfy one of the assumptions of our theory and, consequently, one should not expect an accurate result. To examine a case for which this requirement is satisfied, for example, one can set $\phi = \pi/5$, $n = 2$, and $R = 10$, which means $h/L \sim 1/6$ and $h/R = 1/10$. Experience shows that for cases in which $h/L > h/R$, which is typical for engineering practice, this condition is satisfied and reasonable

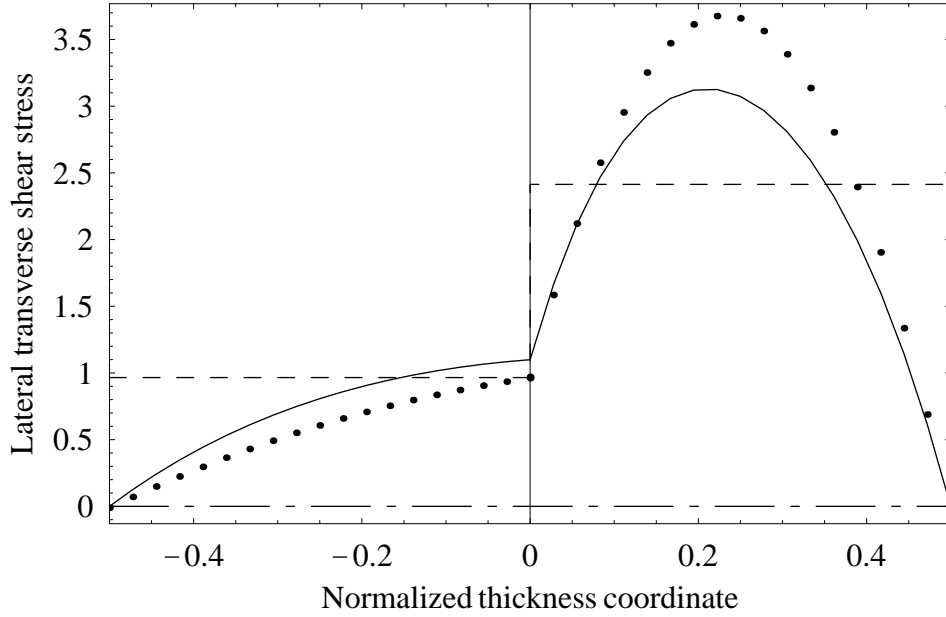


Fig. 3. Distribution of the 3-D stress σ_{13} through the thickness (case 1)

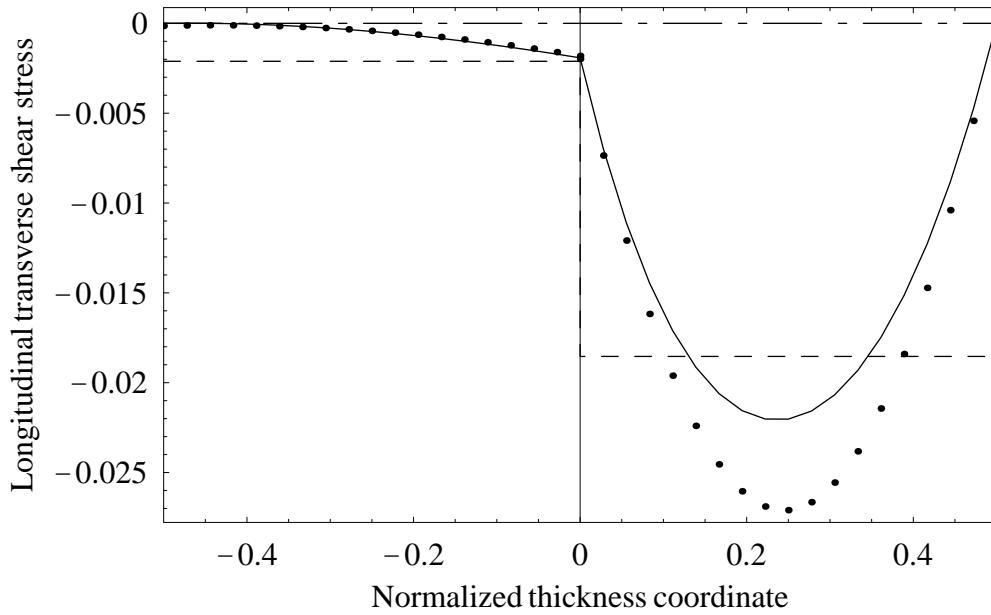


Fig. 4. Distribution of the 3-D stress σ_{23} through the thickness (case 1)

results will be obtained. Note, however, that this does not appear to be a strict requirement for using the present theory, unless one needs accurate transverse normal strain or stress. For the same layup (case 2), the out-of-plane stress components are shown in Figs. 6, 7 and 8. In this case one can see that the present theory predicts the transverse normal stress pretty well.

The beauty of the present theory is that one can easily analyze laminated shells with multiple layers without increasing the computational effort of a standard

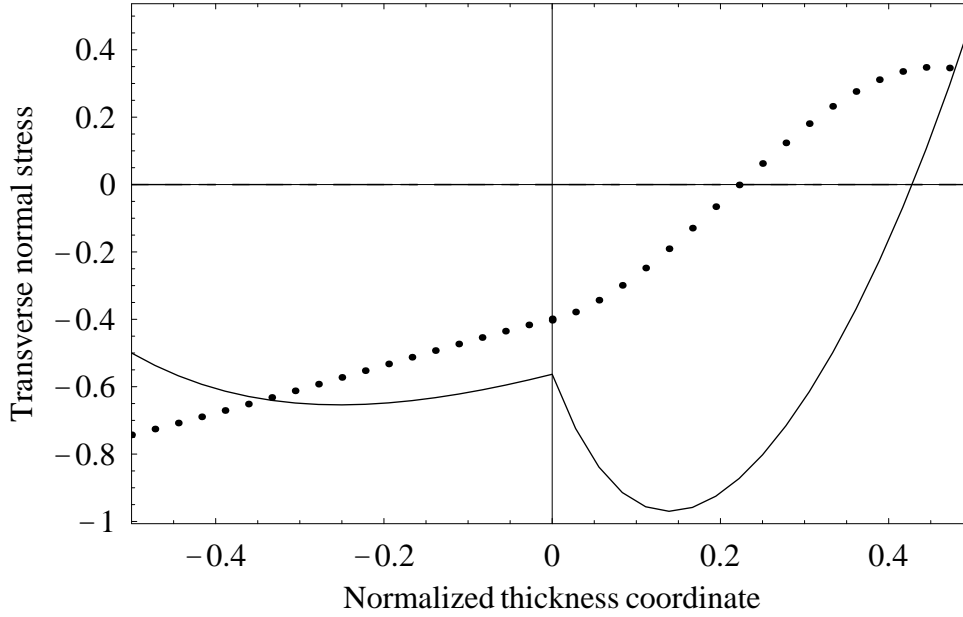


Fig. 5. Distribution of the 3-D stress σ_{33} through the thickness (case 1)

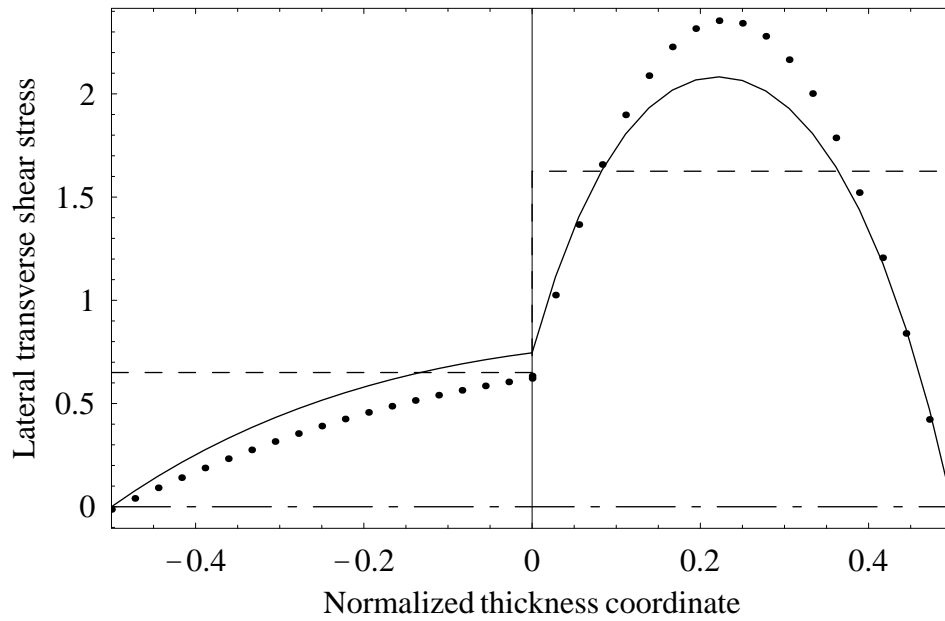


Fig. 6. Distribution of the 3-D stress σ_{13} through the thickness (case 2)

2-D solver since the difficulty of multiple layers is handled in our modeling process. This step is very efficiently solved by a 1-D finite element method in VAPAS and allows accurate results to be obtained without undertaking an expensive 3-D or layer-wise solution. To illustrate this, we use VAPAS to analyze cases 3 and 4. The results are shown in Figs. 9 – 11 and 12 – 14 where excellent correlation is obtained, as expected.

The in-plane quantities are not shown here for the sake of saving space. The

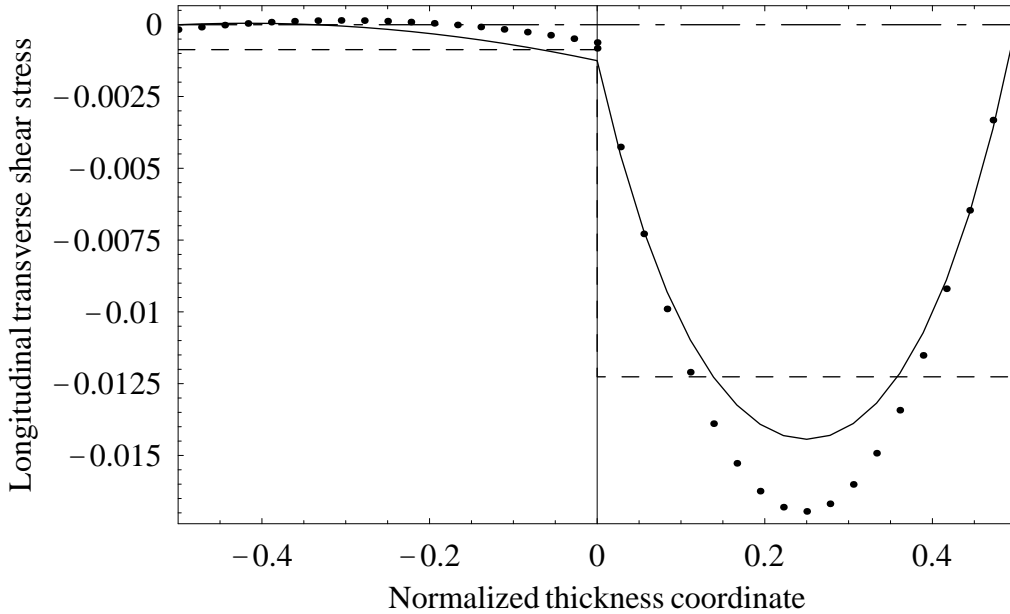


Fig. 7. Distribution of the 3-D stress σ_{23} through the thickness (case 2)

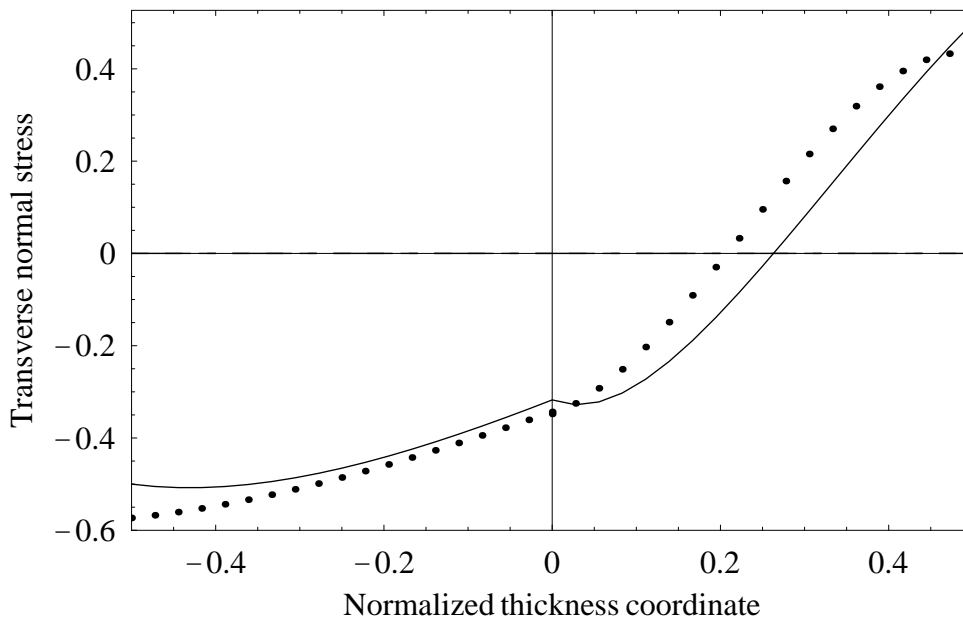


Fig. 8. Distribution of the 3-D stress σ_{33} through the thickness (case 2)

prediction of the present theory for the in-plane quantities is slightly better than that of FOSDT. Note, however, that our FOSDT results are based on the shear stiffness matrix G obtained from VAPAS since the traditional FOSDT has no inherent means to obtain the shear stiffness coefficients for composite shells. The in-plane quantities will converge to those of CLST if h/R is sufficiently small.

The power of the present theory is clearly exhibited in the excellent agreement

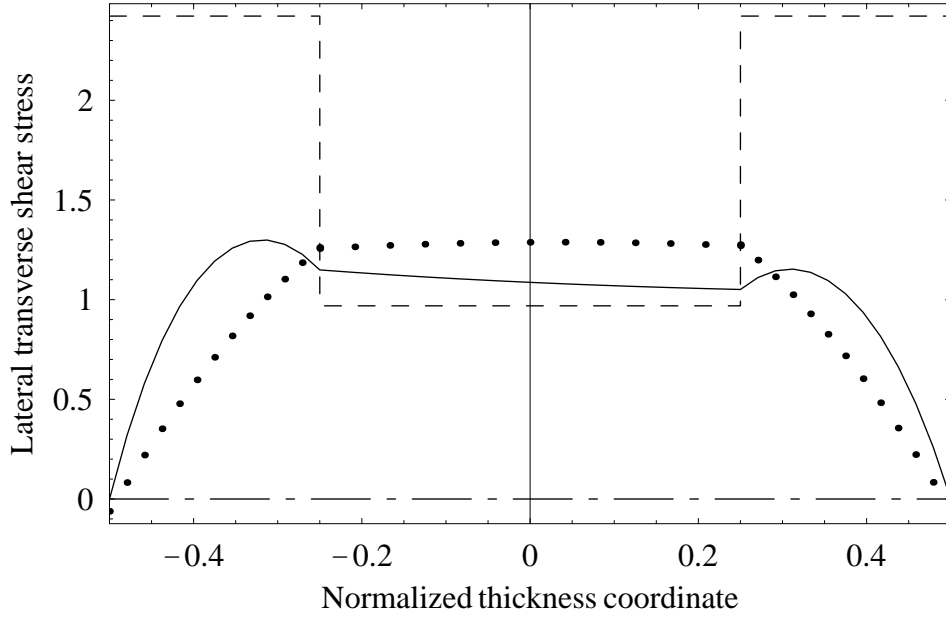


Fig. 9. Distribution of the 3-D stress σ_{13} through the thickness (case 3)

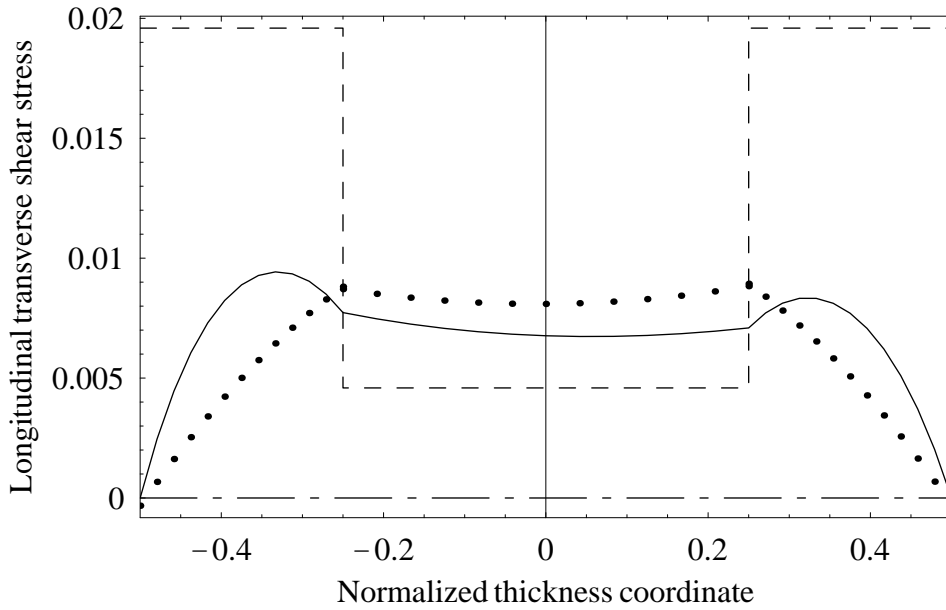


Fig. 10. Distribution of the 3-D stress σ_{23} through the thickness (case 3)

with exact 3-D solutions. Indeed, even though there are more layers in this example, the agreement is still excellent. This clearly proves that one can use VAPAS along with a standard 2-D shell solver for modeling laminated shells confidently to get excellent accuracy with much less computational effort than required by 3-D analysis or by layerwise theories.

It is to be expected that the present theory is far better than *ad hoc* models. Indeed, mathematically, the accuracy of the present theory should be compara-

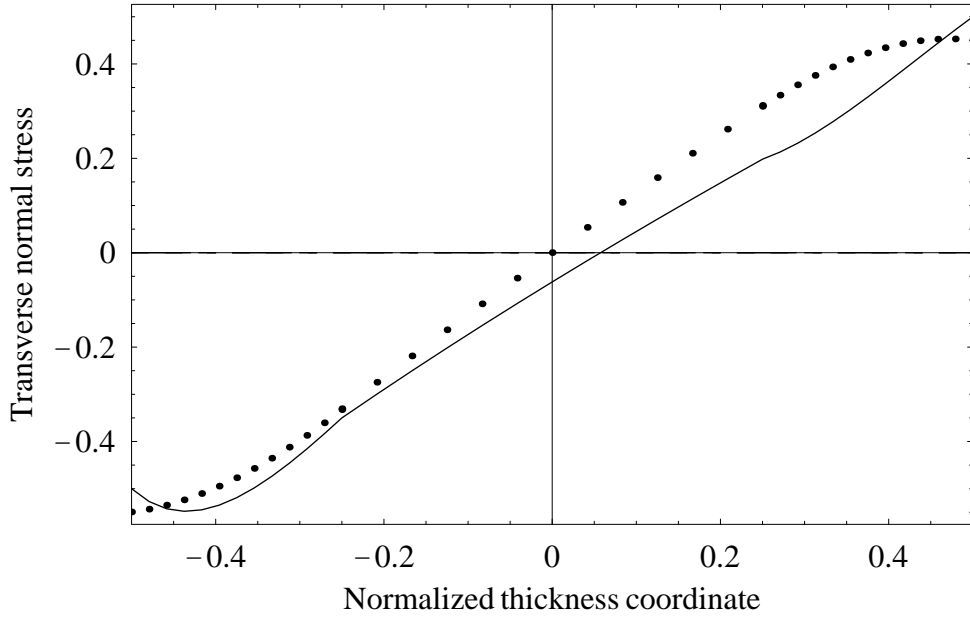


Fig. 11. Distribution of the 3-D stress σ_{33} through the thickness (case 3)

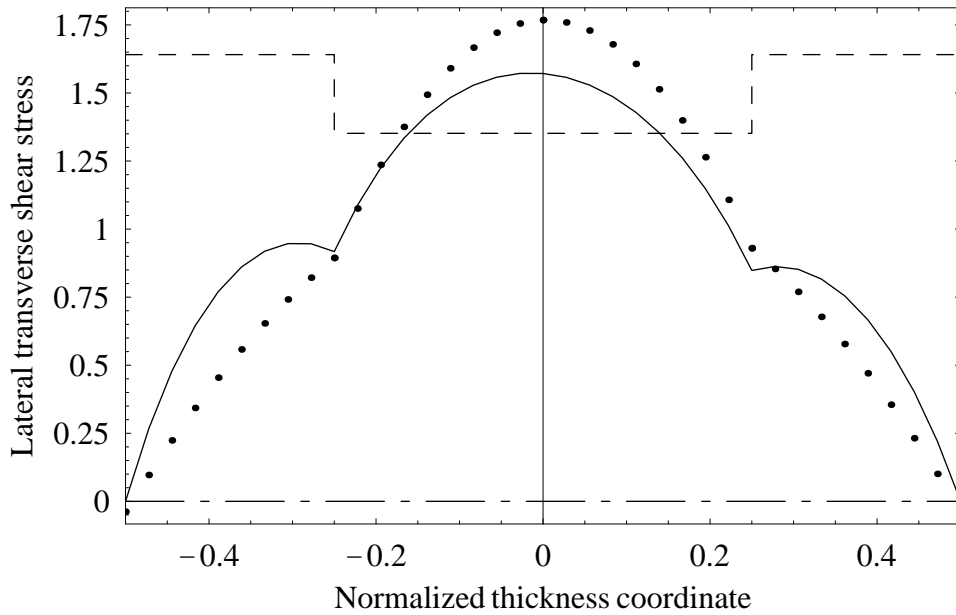


Fig. 12. Distribution of the 3-D stress σ_{13} through the thickness (case 4)

ble to that of a layer-wise shell theory with assumed in-plane displacements as layer-wise cubic polynomials of the thickness direction and transverse displacement as a layer-wise fourth-order polynomial. However, the present theory is still an equivalent single-layer theory, and the computational requirement is thus much less than layer-wise theories. Moreover, it is not necessary to use integration through the thickness of the 3-D equilibrium equations to get the transverse normal and transverse shear strain and stress results presented herein.

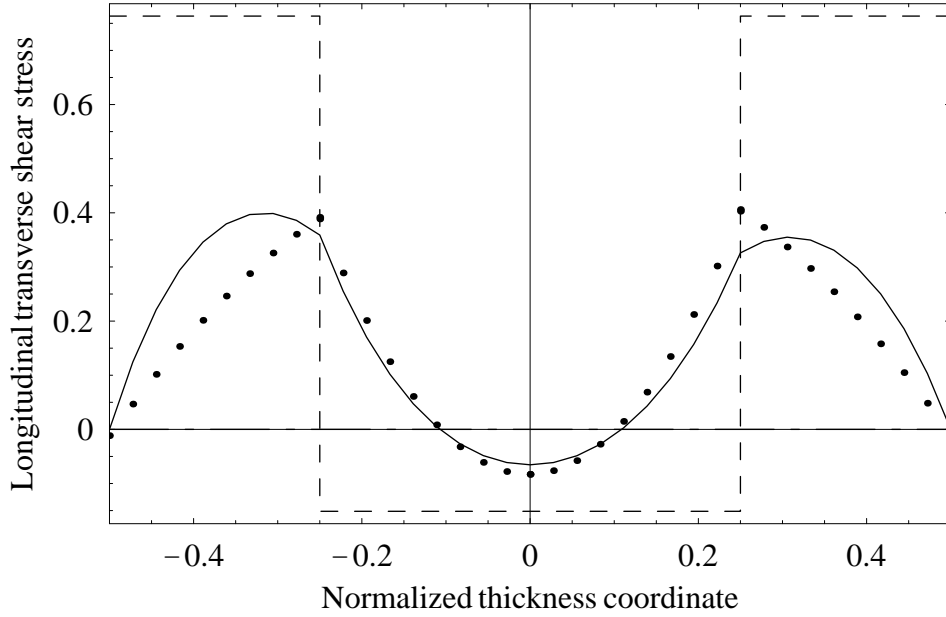


Fig. 13. Distribution of the 3-D stress σ_{23} through the thickness (case 4)

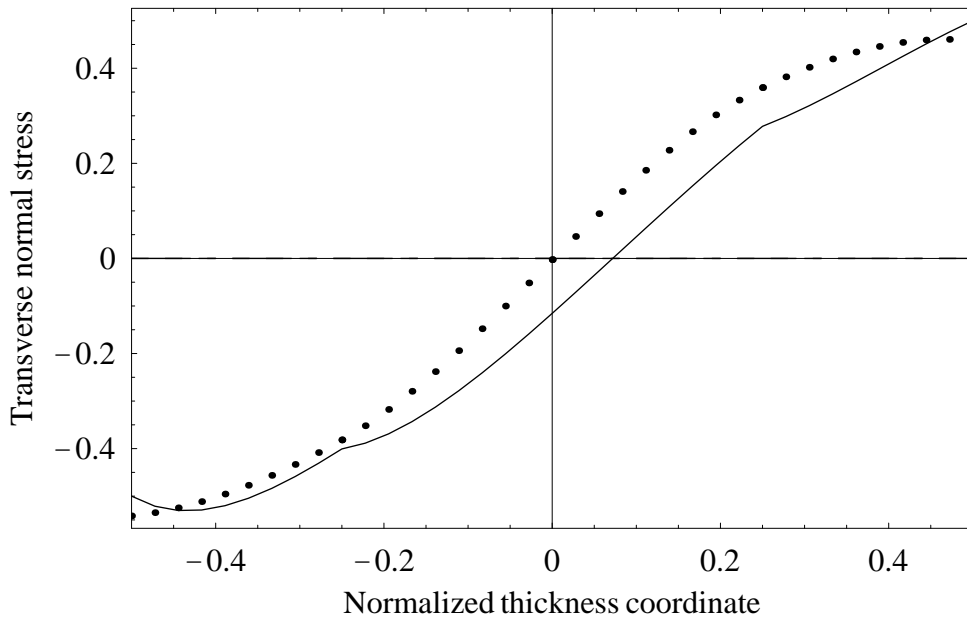


Fig. 14. Distribution of the 3-D stress σ_{33} through the thickness (case 4)

7 Conclusion

A complete Reissner-Mindlin theory for composite laminated shells has been developed from geometrically nonlinear 3-D elasticity. The original 3-D elasticity problem is posed for shells in which each layer is made with a monoclinic material and is formulated in an intrinsic form that is suitable for both geometrically nonlinear as well as linear shell theories. The unknown

3-D warping functions are obtained asymptotically by using the Variational Asymptotic Method and the principle of minimum total potential energy, a procedure which is systematic and easy to apply iteratively. All the potential energy terms are included in the formulation instead of only the strain energy as presented in Ref. [1]. Hence, the resulting 2-D constitutive law includes also a bilinear load-related term in addition to the quadratic terms associated with generalized 2-D strains. The resulting theory is as close as possible to asymptotical correctness by virtue of an optimization procedure.

Although the resulting shell theory is of the form of an equivalent single-layer, first-order shear deformation theory, the 3-D displacement, strain and stress are recovered in closed form with accuracy comparable to that from higher-order, layer-wise shell theories that have many more degrees of freedom. The theory has been implemented in the computer program VAPAS, which calculates the generalized 2-D stiffness matrices (A , B , D , G), the load-related term (F), as well as recovering the 3-D field variables for nonlinear shell problems. Since VAPAS must only solve a 1-D problem, it executes very rapidly, enabling these accurate recovery relations to be cheaply embedded in standard shell finite element codes.

Acknowledgements

This research is supported by the Air Force Office of Scientific Research, USAF, under grant F49620-01-1-0038 (Maj. William M. Hilbun, technical monitor). The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsement, either expressed or implied, of AFOSR or the U.S. Government.

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