On Timoshenko-like Modeling of Initially Curved and Twisted Composite Beams

Wenbin Yu[*], **Dewey H. Hodges**[†], **Vitali Volovoi**[‡],
Georgia Institute of Technology, Atlanta, Georgia
and **Carlos E. S. Cesnik**[§]
Massachusetts Institute of Technology, Cambridge, Massachusetts

Abstract

A generalized, finite-element-based, cross-sectional analysis for nonhomogenous, initially curved and twisted, anistropic beams is formulated from geometrically nonlinear, three dimensional (3-D) elasticity. The 3-D strain field is formulated based on the concept of decomposition of the rotation tensor and is given in terms of 1-D generalized strains and a 3-D warping displacement that is obtained from the formulation, not assumed. The warping is found in terms of the 1-D strains via the variational asymptotic method. In this paper a Timoshenko-like model is presupposed for a beam with cross-sectional characteristic length h, wavelength of deformation given by ℓ, and the magnitude of the radius of initial curvature and/or twist is taken to be of the order R. First, a solution for the asymptotically correct refinement of classical anisotropic beam theory for initially curved and twisted beams through $O(h^2/R^2)$ is obtained. Next, the $O(h^2/\ell^2)$ correction is computed. It is known that an asymptotically correct Timoshenko-like theory is not capable of capturing all the $O(h^2/\ell^2)$ corrections for generally anisotropic beams. However, if all the $O(h^2/\ell^2)$ terms are known, then the corresponding Timoshenko-like theory is uniquely defined. Numerical results are presented to illustrate the trends of the various classical (extension-twist, bending-twist, and extension-bending) and non-classical couplings (extension-shear, bending-shear, and shear-torsion) as the initial twist and curvatures are varied.

[*]Graduate Research Assistant, School of Aerospace Engineering.
[†]Professor, School of Aerospace Engineering.
[‡]Research Engineer, School of Aerospace Engineering.
[§]Associate Professor, Department of Aeronautics and Astronautics. Presently, Associate Professor, Department of Aerospace Engineering, University of Michigan, Ann Arbor, Michigan.

Introduction

A beam is a flexible body for which one dimension is much larger than the other two. To take advantage of this geometric feature without a loss of accuracy, one has to capture the behavior associated with the two dimensions eliminated in the model. To complicate matters, in composite beams there may be elastic couplings among all the forms of deformation. Moreover, the in- and out-of-plane warping displacements may be coupled. The stiffness model is strongly affected by these complications. In recent years, research has yielded significant advances in analyzing composite beams. Hodges (1990b) gives a complete review of such literature prior to 1988. Cesnik and Hodges (1997) and Popescu and Hodges (1999b) give additional, more up-to-date reviews. To the authors' knowledge, the most recent review of the literature associated with composite beam modeling can be found in Jung *et al.* (1999) and Volovoi *et al.* (1999).

A classical theory accounts for extension, torsion, and bending in two directions. Stemming from the variational asymptotic method developed by Berdichevsky (1976), the development of cross-sectional analyses for classical modeling of nonhomogeneous, anisotropic beams is well understood. There are at least two unique features of this approach. First, it is based on 3-D nonlinear theory, and second it provides the basis for both the (typically) linear, 2-D cross-sectional analysis and the 1-D nonlinear equations. It has now been applied to prismatic beams in Hodges *et al.* (1992), initially curved and twisted beams in Cesnik *et al.* (1996) and Cesnik and Hodges (1997), beams with oblique cross-sectional planes in Popescu *et al.* (2000) and modeling of the trapeze effect in Popescu and Hodges (1999a).

Classical theory is adequate in many situations, namely when the beam is slender, is not a thin-walled open section, and undergoes motions with large wavelength (*i.e.*, low-frequency modes of vibration). However, a refined theory is required for high accuracy in other situations. There are at least three types of refinements: (1) the Timoshenko refinement, needed for short-wavelength modes associated with transverse shear effects; (2) the Vlasov refinement, typically needed for thin-walled, open-section beams; (3) a general refined theory, in which new degrees of freedom are chosen based on criteria that are problem dependent (high frequency vibrations, sandwich beams, etc.). Only the first type of refinement is considered in this paper.

This type of theory was first developed for beams with arbitrary cross-sectional geometry and material properties in Giavotto *et al.* (1983). Therein, in accordance with Saint-Venant's principle, only the so-called "central" solutions are directly sought after. These are the ones with a stress distribution that only has a polynomial dependence on the axial coordinate. This way all the decaying solutions are explicitly removed from the problem and six canonical problems were solved in order to obtain a 6×6 cross-sectional flexibility matrix. The procedure has been further refined in Borri and Merlini (1986) where the more elegant concept of "intrinsic warping" was introduced and used instead of solving six canonical problems. The ramifications of this refinement on final values of beam model are not obvious, but no differences in the results have been reported. The approach used in Borri and Merlini (1986) was later applied to initially curved and twisted beams as well in Borri *et al.* (1992). This work led to the computer code Anisotropic Beam Analysis (ANBA) which has been the standard by which other analyses have been judged since the early 1980s. The English version of this code we have used for comparison is an implementation of Giavotto *et al.* (1983), which applies only to prismatic beams and created for internal use at Georgia Tech by Prof. O. Bauchau. This version was called Nonhomogeneous, Anisotropic Beam Section Analysis (NABSA).

It is useful to point out the main differences between the work found in Borri and Merlini (1986); Borri *et al.* (1992) and the present work. In the earlier work no asymptotic assessment of the 3-D strain energy was made. Instead the Timoshenko-like form of the 1-D strain energy was directly fit to the 3-D strain energy. The mismatch between the 3-D energy and its 1-D representation was reconciled using a semi-discretization in which the axial coordinate appears as a low-order polynomial and the cross-sectional coordinates are treated using a finite element discretization. Furthermore, those authors made use of a concept they called "intrinsic warping" that effectively redefines the 1-D variables. There is no known basis for extending the earlier work to include end effects, such as the Vlasov phenomenon. Finally, this analysis is essentially linear in all aspects, and the trapeze effect was added by consideration of the axial stress as initial stress.

On the other hand, the asymptotic approach reveals that Timoshenko-like terms in the energy are $O(h^2/\ell^2)$, and it is only those terms that are being "fit" into the Timoshenko-like form of the 1-D strain energy. As a result, no introduction of intrinsic warping is required, and the constraints that are imposed on the warping do not redefine the 1-D variables. It has to be noted that the asymptotic approach naturally includes the treatment of nonlinearities and provides a rigorous basis for extension to refinements of other kinds. It is remarkable that, as shown below, both approaches lead to results that are very close to each other numerically.

The variational asymptotic method was applied to the Timoshenko-like modeling of prismatic isotropic beams by Berdichevsky and Kvashnina (1976). Their work was extended to treat composite, prismatic beams in Popescu and Hodges (1999b). However, this work invoked some compromises. First, a least squares technique was used to calculate the stiffness matrix, which turns out to be unnecessary as shown in this paper. Second, there is a slight inconsistency in the equations in that work (which has turned out to have little or no effect on the results for most cases). Finally, the initial curvature and twist effects, although treated for classical theory in Cesnik and Hodges (1997), were not yet developed for the refined theory. Most important is that now the engineering software – the Variational Asymptotic Beam Sectional Analysis (*VABS*) – has been extended to include all the capability described here. It can be used to perform a generalized cross-sectional modeling approach to obtain a 6×6 stiffness matrix along with the interior solution of the warping for initially curved and twisted composite beams with arbitrary choice of reference line, which is sufficient for most engineering application.

Beam Kinematics

To analyze a beam, a reference line r should be specified (Fig. 1). This choice is arbitrary. A typical cross section could be described as a prescribed domain s with h as its characteristic size. Any material point of the beam in 3-D space can be located by a position vector $\mathbf{r}$, which is specified by the beam axial coordinate x_1 along r and cross-sectional Cartesian coordinates (x_2, x_3) embedded in a chosen reference cross section. At each point along r an orthogonal reference triad $\mathbf{b}_i$ is introduced such that $\mathbf{b}_\alpha$ is tangent to x_α and $\mathbf{b}_1 = \mathbf{b}_2 \times \mathbf{b}_3$. (Here and throughout all the paper, Greek indices assume values 2 and 3 while Latin indices assume 1, 2, and 3. Repeated indices are summed over their range except where explicitly indicated.) The vectors $\mathbf{b}_i$ have the properties that

$$\mathbf{b}_i \cdot \mathbf{b}_j = \delta_{ij} \tag{1}$$

where δ_{ij} is the Kronecker symbol.

Now the spatial position vector $\bar{\mathbf{r}}$ of any point in the cross section can be written as:

$$\hat{\mathbf{r}}(x_1, x_2, x_3) = \mathbf{r}(x_1) + x_\alpha \mathbf{b}_\alpha(x_1) \tag{2}$$

where $\mathbf{r}$ is the position vector of the points of the reference line, $\mathbf{r}' = \mathbf{b}_1$ and $(\ )'$ means the derivative to x_1.

When the beam deforms, the triads $\mathbf{b}_i$ rotate to coincide with new triads $\mathbf{B}_i$. Here $\mathbf{B}_1$ is not tangent to x_1 if shear deformation is considered. For the purpose of making the derivation more convenient, we introduce another triad $\mathbf{T}_i$ associated with the deformed beam (see Fig. 2), with $\mathbf{T}_1$ tangent to the deformed beam reference line and $\mathbf{T}_\alpha$ determined by a rotation about $\mathbf{T}_1$. Obviously, the difference in the orientations of $\mathbf{T}_i$ and $\mathbf{B}_i$ is due to the small rotations associated with transverse shear deformation. The relationship between these two basis vectors can be expressed as

$$\left\{ \begin{array}{c} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \end{array} \right\} = \begin{bmatrix} 1 & -2\gamma_{12} & -2\gamma_{13} \\ 2\gamma_{12} & 1 & 0 \\ 2\gamma_{13} & 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} \mathbf{T}_1 \\ \mathbf{T}_2 \\ \mathbf{T}_3 \end{array} \right\} \tag{3}$$

where $2\gamma_{12}$ and $2\gamma_{13}$ are the small angles caused by the shear deformation.

One can represent the position vector of any particle in the deformed beam which had position $\hat{\mathbf{r}}$ in the undeformed beam as

$$\hat{\mathbf{R}}(x_1, x_2, x_3) = \mathbf{R}(x_1) + x_\alpha \mathbf{T}_\alpha(x_1) + w_i(x_1, x_2, x_3)\mathbf{T}_i(x_1) \tag{4}$$

where $\mathbf{R}$ is the position vector to a point on the reference line of the deformed beam, and w_i are the components of warping. Note in this formulation, that we choose $\mathbf{T_1}$ to be tangent to the deformed beam reference line, which means we classify the transverse shear deformation as part of the warping field. Within the framework of small strain this does not introduce any additional approximation or result in any loss of information. We note that the transverse shear strain measures are typically one order higher in h/ℓ than the classical 1-D strain measures. The 1-D generalized strain measures of the classical theory can be expressed as:

$$\varepsilon = [\gamma_{11} \quad \kappa_1 \quad \kappa_2 \quad \kappa_3]^T \tag{5}$$

Here, consistent with the geometrically-exact framework of Hodges (1990a), the "moment-strain" measures κ_i are defined based on the changes along x_1 of the triad $\mathbf{T}_i$, viz.,

$$\frac{d\mathbf{T}_i}{dx_1} = (k_j + \kappa_j)\mathbf{T}_j \times \mathbf{T}_i \tag{6}$$

where k_1 is the initial twist and k_α are the initial curvatures such that

$$\frac{d\mathbf{b}_i}{dx_1} = k_j \mathbf{b}_j \times \mathbf{b}_i \tag{7}$$

Because of the special choice of triad $\mathbf{T}_i$, the 1-D "force-strain" measures associated with shear deformation are zero. In order to consider the case in which the 1-D transverse shear measures are not zero, another set of moment-strain measures associated with the $\mathbf{B}_i$ basis will be introduced below, denoted by β_i (see Eq. 34) where

$$\frac{d\mathbf{B}_i}{dx_1} = (k_j + \beta_j)\mathbf{B}_j \times \mathbf{B}_i \tag{8}$$

One should note that Eq. (4) is four times redundant because of the way warping was introduced. One could impose four appropriate constraints on the displacement field to remove the redundancy. The four constraints applied here are

$$\langle\langle w_i \rangle\rangle = 0 \tag{9}$$

$$\langle\langle x_2\, w_3 - x_3\, w_2 \rangle\rangle = 0 \tag{10}$$

where the notation $\langle\langle\ \rangle\rangle$ means integration over the reference cross section. The implication of the Eq. (10) is that warping does not contribute to the rigid displacements of the cross section. This leads to the 1-D displacement variables for extension and bending that have easily identifiable geometric meanings: they correspond to the measure numbers in the $\mathbf{b}_i$ basis of the average displacement of the cross section. The torsional rotation variable is the average rotation of the cross section about $\mathbf{B}_1$.

3-D Formulation

Based on the concept of decomposition of the rotation tensor in Danielson and Hodges (1987), the Jauman-Biot-Cauchy strain components for small local rotation are given by

$$\Gamma_{ij} = \frac{1}{2}(F_{ij} + F_{ji}) - \delta_{ij} \tag{11}$$

where F_{ij} is the mixed-basis component of the deformation gradient tensor such that

$$F_{ij} = \mathbf{T}_i \cdot \mathbf{G}_k \mathbf{g}^k \cdot \mathbf{b}_j \tag{12}$$

with the base vectors defined as

$$\begin{aligned} \mathbf{g}_i &= \frac{\partial \hat{\mathbf{r}}}{\partial x_i} \\ \mathbf{G}_i &= \frac{\partial \hat{\mathbf{R}}}{\partial x_i} \\ \mathbf{g}^i &= \frac{e_{ijk}\mathbf{g_j} \times \mathbf{g_k}}{2\sqrt{g}} \end{aligned} \tag{13}$$

where g is the determinant of the metric tensor for the undeformed geometry $g_{ij} = \det(\mathbf{g}_i \cdot \mathbf{g}_j)$, and e_{ijk} are the components of the permutation tensor in Cartesian coordinate system.

Discarding the product of the warping and 1-D generalized strains (both are of the order of the strain), one can express the 3-D strain field as

$$\Gamma = \Gamma_h\ w + \Gamma_\varepsilon\ \varepsilon + \Gamma_R\ w + \Gamma_l\ w' \tag{14}$$

where $\Gamma = [\ \Gamma_{11}\ \ 2\Gamma_{12}\ \ 2\Gamma_{13}\ \ \Gamma_{22}\ \ 2\Gamma_{23}\ \ \Gamma_{33}\ ]^T$, $w = [\ w_1\ \ w_2\ \ w_3\ ]^T$. All of the operators in Eq. (14) are defined as follows:

$$\Gamma_h = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\partial}{\partial x_2} & 0 & 0 \\ \frac{\partial}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & 0 & \frac{\partial}{\partial x_3} \end{bmatrix} \tag{15}$$

$$\Gamma_\epsilon = \frac{1}{\sqrt{g}} \begin{bmatrix} 1 & 0 & x_3 & -x_2 \\ 0 & -x_3 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

$$\Gamma_R = \frac{1}{\sqrt{g}} \begin{bmatrix} \widetilde{k} + I_3 k_1 \left(x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} \right) \\ O_3 \end{bmatrix} \tag{17}$$

$$\Gamma_l = \frac{1}{\sqrt{g}} \begin{bmatrix} I_3 \\ O_3 \end{bmatrix} \tag{18}$$

where O_3 is a 3×3 null matrix, I_3 is a 3×3 identity matrix, and the operator $\widetilde{\ }$ is defined as $\widetilde{\bullet} = -e_{ijk} \bullet_k$. This form of strain field is of great importance, because it is now linear in ε, w and its derivatives.

The strain energy of the cross section or the strain energy density of the beam may be written as

$$U = \frac{1}{2} \left\langle \Gamma^T \mathcal{D} \, \Gamma \right\rangle \tag{19}$$

The notation $\langle \bullet \rangle = \int_s \bullet \sqrt{g}\ dx_2\, dx_3$ is used throughout the paper, where $\mathcal{D}$ is the 6×6 symmetric material matrix in the $\mathbf{b}_i$ basis. Now the basic 3-D elastic beam problem is to minimize the functional Eq. (19) subject to the constraints in Eqs. (9) and (10) to find the unknown warping function.

Solution for the Warping

The behavior of an elastic body is completely determined by its energy. Hence, to derive 1-D beam theory, one has to reproduce the 3-D energy in terms of 1-D quantities. This dimensional reduction cannot be done exactly. The Variational Asymptotic Method (*VAM*), Berdichevsky (1976), is a powerful mathematical method that can be used to find the 1-D energy which approximates the 3-D energy as closely as possible. To solve this problem via the *VAM*, small parameters should be first identified. While it is true that the strain is a small parameter, because we are limiting the present discussion to physically linear theory, it is appropriate to discard terms from the strain energy that are of higher order than quadratic in the strain. It is known for a beam that $h/\ell < 1$ and $h/R < 1$, where ℓ is the characteristic wavelength of deformation along the beam axial coordinate and R is the characteristic radius of initial curvature/twist of the beam. For convenience, and because it is sufficient for the purposes of developing a Timoshenko-like theory for initially curved and twisted beams to do so, we assume that ℓ and R are of the same order. Then it is necessary only to expand all unknown functions in asymptotic series of h; *VAM* can then be used to asymptotically reduce the original 3-D problem to a 1-D problem.

In order to deal with arbitrary cross-sectional geometry and anisotropic materials, one may turn to a numerical approach to find the stationary value of the functional. The warping field can be discretized as

$$w(x_1, x_2, x_3) = S(x_2, x_3)\, V(x_1) \tag{20}$$

with $S(x_2, x_3)$ representing the element shape function and V as a column matrix of the nodal values of the warping displacement over the cross section. Substituting Eq. (20) back into Eq. (19), one obtains

$$\begin{aligned} 2U = {} & V^T E\, V + 2\, V^T \left(D_{h\epsilon}\, \epsilon + D_{hR} V + D_{hl} V'\right) + \epsilon^T D_{\epsilon\epsilon}\, \epsilon \\ & + V^T D_{RR} V + V'^T D_{ll} V' + 2 V^T D_{R\epsilon} \epsilon + 2 V'^T D_{l\epsilon} \epsilon + 2 V^T D_{Rl} V' \end{aligned} \tag{21}$$

where the newly introduced matrices are defined as

$$\begin{aligned} E &= \left\langle [\Gamma_h\, S]^T\ \mathcal{D}\ [\Gamma_h\, S] \right\rangle & D_{h\varepsilon} &= \left\langle [\Gamma_h\, S]^T\ \mathcal{D}\ [\Gamma_\varepsilon\, S] \right\rangle \\ D_{hR} &= \left\langle [\Gamma_h\, S]^T\ \mathcal{D}\ [\Gamma_R\, S] \right\rangle & D_{hl} &= \left\langle [\Gamma_h\, S]^T\ \mathcal{D}\ [\Gamma_l\, S] \right\rangle \\ D_{\varepsilon\varepsilon} &= \left\langle [\Gamma_\varepsilon\, S]^T\ \mathcal{D}\ [\Gamma_\varepsilon\, S] \right\rangle & D_{RR} &= \left\langle [\Gamma_R\, S]^T\ \mathcal{D}\ [\Gamma_R\, S] \right\rangle \\ D_{ll} &= \left\langle [\Gamma_l\, S]^T\ \mathcal{D}\ [\Gamma_l\, S] \right\rangle & D_{R\varepsilon} &= \left\langle [\Gamma_R\, S]^T\ \mathcal{D}\ [\Gamma_\varepsilon\, S] \right\rangle \\ D_{l\varepsilon} &= \left\langle [\Gamma_l\, S]^T\ \mathcal{D}\ [\Gamma_\varepsilon\, S] \right\rangle & D_{Rl} &= \left\langle [\Gamma_R\, S]^T\ \mathcal{D}\ [\Gamma_l\, S] \right\rangle \end{aligned} \tag{22}$$

These matrices carry information on the material properties as well as the geometry of a given cross section.

To use the *VAM*, one has to find the leading terms of the functional according to different orders. For the zeroth-order approximation, the leading terms of Eq. (21) are

$$2U_0 = V^T E\, V + V^T D_{h\varepsilon}\, \varepsilon + \varepsilon^T D_{\varepsilon\varepsilon}\, \varepsilon \tag{23}$$

Minimizing the functional in Eq. (23) subject to the constraint in Eqs. (9) and (10), one finds the zeroth approximation of the warping to be

$$V = V_0 = \hat{V}_0 \varepsilon \tag{24}$$

Plugging Eq. (24) back into the energy expression, Eq. (23), one can obtain the energy up to the order of $O(\varepsilon^2)$ as

$$2U_0 = \varepsilon^T (\hat{V}_0^T D_{h\varepsilon} + D_{\varepsilon\varepsilon})\varepsilon \tag{25}$$

We now perturb the unknown warping field as a series in the parameter h, so that

$$V = V_0 + hV_1 + h^2 V_2 + O(h^3) \tag{26}$$

Substituting Eq. (26) into Eq. (21), one can prove that it is not necessary to calculate $h^2 V_2$ and higher-order quantities in order to obtain an asymptotically correct energy expression through $O(h^2\varepsilon^2)$. Thus, the energy can be written as

$$\begin{aligned} 2U_1 = {} & \epsilon^T (\hat{V}_0^T D_{h\epsilon} + D_{\epsilon\epsilon})\epsilon + 2(V_0^T D_{hR} V_0 + V_0^T D_{hl} V_0' + V_0^T D_{R\epsilon}\epsilon + V_0'^T D_{l\epsilon}\epsilon) \\ & + V_1^T E V_1 + 2V_1^T (D_{hR} V_0 + D_{hR}^T V_0 + D_{R\epsilon}\epsilon) + 2V_1^T D_{hl} V_0' + 2V_0^T D_{hl} V_1' \\ & + 2V_1'^T D_{l\epsilon}\epsilon + V_0^T D_{RR} V_0 + 2V_0^T D_{Rl} V_0' + V_0'^T D_{ll} V_0' \end{aligned} \tag{27}$$

After integrating by parts, the second-order leading terms (without the constant terms) are

$$2U_2 = V_1^T E V_1 + 2V_1^T D_R \varepsilon + 2V_1^T D_S \varepsilon' \tag{28}$$

where

$$D_R = D_{hR}\hat{V}_0 + D_{hR}^T \hat{V}_0 + D_{R\varepsilon} \tag{29}$$

$$D_S = D_{hl}\hat{V}_0 - D_{hl}^T \hat{V}_0 - D_{l\varepsilon} \tag{30}$$

From here, one could solve for the first-order approximation of warping, obtaining

$$V_1 = V_{1R}\varepsilon + V_{1S}\varepsilon' \tag{31}$$

Using Eq. (31), the second-order asymptotically correct energy can now be obtained from

Eq. (27) as

$$2U = \varepsilon^T A \varepsilon + 2\varepsilon^T B \varepsilon' + \varepsilon'^T C \varepsilon' + 2\varepsilon^T D \varepsilon'' \tag{32}$$

where

$$\begin{aligned}
A &= \hat{V}_0^T D_{h\epsilon} + D_{\epsilon\epsilon} + \hat{V}_0^T (D_{hR} + D_{hR}^T + D_{RR}) \hat{V}_0 + \hat{V}_0^T D_{R\epsilon} + D_{R\epsilon}^T V_0 + D_R^T V_{1R} \\
B &= \hat{V}_0^T D_{hl} \hat{V}_0 + D_{l\epsilon}^T \hat{V}_0 + \hat{V}_0^T D_{hl} V_{1R} + D_{l\epsilon}^T V_{1R} + \hat{V}_0^T D_{Rl} \hat{V}_0 \\
&\quad + \frac{1}{2}(D_R^T V_{1S} + V_{1R}^T D_{hl} \hat{V}_0 + V_{1R}^T D_{hl}^T \hat{V}_0 + V_{1R}^T D_{l\epsilon}) \\
C &= \hat{V}_0^T D_{hl}^T V_{1S} + V_{1S}^T D_{hl}^T \hat{V}_0 + V_{1S}^T D_{l\epsilon} + \hat{V}_0^T D_{ll} \hat{V}_0 \\
D &= (D_{l\epsilon}^T + \hat{V}_0^T D_{hl}) V_{1S}
\end{aligned} \tag{33}$$

Transformation to Timoshenko-Like Theory

Straightforward use of the strain energy, Eq. (32), is possible as mentioned in Sutyrin (1997); but this involves boundary conditions that are more complicated than necessary. Although this formulation is asymptotically correct up to the second order, it contains derivatives of the classical strain measures. The problem of how to get rid of these undesirable derivatives, while keeping the theory asymptotically correct, has not yet been brought to a completely general conclusion. Berdichevsky and Starosel'skii (1983) used changes of variables to tackle this problem, while Popescu and Hodges (1999b) used the 1-D equilibrium equations to build a relationship between the strains and derivatives of strains. Although the change of variables can yield an elegant strain energy of Timoshenko-like form, the kinematic meaning of transverse shear strains is lost. Hence, equilibrium equations are used in the present work.

First, one has to change the asymptotically correct formulation to be expressed by the strain measures of Timoshenko theory. Recalling the deformed beam system $\mathbf{B}_i$, we have already mentioned that $\mathbf{B}_1$ is not tangent to x_1 because of shear deformation (see Fig. 2). Based on the relation between $\mathbf{T}_i$ and $\mathbf{B}_i$ and assuming small strains, one can derive a kinematic identity between classical strains in the $\mathbf{T}_i$ system and the Timoshenko strains in the $\mathbf{B}_i$ system, similar to the manner in which this was done in Popescu and Hodges (1999b). Thus,

$$\varepsilon = \epsilon + Q\,\gamma' + P\,\gamma \tag{34}$$

where ε is the same as mentioned before and $\epsilon = \lfloor\ \gamma_{11} \quad \beta_1 \quad \beta_2 \quad \beta_3\ \rfloor^T$ represents the 1-D generalized strains associated with the $\mathbf{B}_i$ system due to extension, torsion, and bending.

The matrices Q and P are given by

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 0 \\ k_2 & k_3 \\ -k_1 & 0 \\ 0 & -k_1 \end{bmatrix} \tag{35}$$

with $\gamma = [2\gamma_{12} \;\; 2\gamma_{13}]^T$ as the column matrix of transverse shear strain measures associated with the $\mathbf{B}_i$ basis. Definitions of γ_{11} and γ in a geometrically-exact framework are given in Hodges (1990a).

Substituting Eq. (34) into Eq. (32) and dropping higher-order terms, one can express the strain energy in terms of the Timoshenko-like beam strain measures as

$$2U_1 = \epsilon^T A\epsilon + 2\epsilon^T AQ\gamma' + 2\epsilon^T AP\gamma + 2\epsilon^T B\epsilon' + \epsilon'^T C\epsilon' + 2\epsilon^T D\epsilon'' \tag{36}$$

The Timoshenko-like strain energy should be written as

$$2U = \epsilon^T X\epsilon + 2\epsilon^T F\gamma + \gamma^T G\gamma \tag{37}$$

Theoretically, if the Timoshenko-like theory is asymptotically correct, Eqs. (36) and (37) should be equivalent. To prove this, one has to get rid of the derivatives in Eq. (36). Using the equilibrium equations is a feasible way to achieve this. Following Hodges (1990a), the nonlinear 1-D equilibrium equations for initially curved and twisted beams without distributed forces, both applied and inertial, can be written as

$$\begin{aligned} &F' + \widetilde{K}F = 0 \\ &M' + \widetilde{K}M + (\widetilde{e_1} + \widetilde{\rho})F = 0 \end{aligned} \tag{38}$$

with

$$\begin{aligned} e_1 &= \lfloor 1 \;\; 0 \;\; 0 \rfloor^T \\ \rho &= \lfloor \gamma_{11} \;\; 2\gamma_{12} \;\; 2\gamma_{13} \rfloor^T \\ K &= k + \beta \end{aligned} \tag{39}$$

Here F is the column matrix of the cross-sectional stress resultant measures in the $\mathbf{B}_i$ basis, and M is the column matrix of the cross-sectional moment resultant measures in the $\mathbf{B}_i$ basis. In our asymptotic analysis, terms of order $\mu\varepsilon^3$ and $\mu\varepsilon^2 h^3$ are neglected in the strain energy which leads to the estimation $\varepsilon = O(h^3)$. Because the strain energy is only asymptotically correct up to the second order of h, nonlinear terms in the equilibrium equations do not affect the strain energy. Therefore, for the purposes of creating the cross-sectional model, Eq. (38) can be simplified to read

$$\left\{\begin{array}{c} F_2' \\ F_3' \end{array}\right\} + D_1 \left\{\begin{array}{c} F_2 \\ F_3 \end{array}\right\} + D_2 \left\{\begin{array}{c} F_1 \\ M_1 \\ M_2 \\ M_3 \end{array}\right\} = 0 \tag{40}$$

$$\left\{\begin{array}{c} F_1' \\ M_1' \\ M_2' \\ M_3' \end{array}\right\} + D_3 \left\{\begin{array}{c} F_1 \\ M_1 \\ M_2 \\ M_3 \end{array}\right\} + D_4 \left\{\begin{array}{c} F_2 \\ F_3 \end{array}\right\} = 0 \tag{41}$$

where

$$D_1 = \left[\begin{array}{cc} 0 & -k_1 \\ k_1 & 0 \end{array}\right]; D_2 = \left[\begin{array}{cccc} k_3 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \end{array}\right]; D_3 = \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & -k_3 & k_2 \\ 0 & k_3 & 0 & -k_1 \\ 0 & -k_2 & k_1 & 0 \end{array}\right] \tag{42}$$

and $D_4 = Q - D_2^T$.

One can express ϵ' and γ' in terms of ϵ and γ from Eqs. (40) and (41) as

$$\epsilon' = N^{-1}(A_3\gamma + A_4\epsilon) \tag{43}$$

$$\gamma' = -G^{-1}[(F^T N^{-1} A_3 + D_1 G + D_2 F)\gamma + (F^T N^{-1} A_4 + D_1 F^T + D_2 X)\epsilon] \tag{44}$$

where

$$A_3 = (FG^{-1}D_1 - D_4)G + (FG^{-1}D_2 - D_3)F \tag{45}$$

$$A_4 = A_3 G^{-1} F^T + (FG^{-1}D_2 - D_3)N \tag{46}$$

$$N = X - FG^{-1}F^T \tag{47}$$

Differentiating the both sides of Eq. (43), one can express ϵ'' in terms of ϵ and γ with the help of Eq. (44) as:

$$\begin{aligned} \epsilon'' &= N^{-1}\left[(FG^{-1}D_2 - D_3)A_3 - A_3G^{-1}(D_1G + D_2F)\right]\gamma \\ &+ N^{-1}\left[(FG^{-1}D_2 - D_3)A_4 - A_3G^{-1}(D_1F^T + D_2X)\right]\epsilon \end{aligned} \tag{48}$$

Substitution of Eqs. (43), (44) and (48) back into Eq. (36), one will get a Timoshenko-like energy expression. The next step is to set this form equal to Eq. (37), after which one can obtain by inspection the following matrix equations

$$
\begin{aligned}
X &= A - 2AQG^{-1}(FN^{-1}A_4 + D_1F^T + D_2X) + 2BN^{-1}A_4 + A_4^TN^{-1}CN^{-1}A_4 \\
&\quad + 2DN^{-1}\left[(FG^{-1}D_2 - D_3)A_4 - A_3G^{-1}(D_1F^T + D_2X)\right] \qquad (49) \\
F &= -AQG^{-1}(FN^{-1}A_3 + D_1G + D_2F) + BN^{-1}A_3 + A_4^TN^{-1}CN^{-1}A_3 + AP \\
&\quad + N^{-1}\left[(FG^{-1}D_2 - D_3)A_3 - A_3G^{-1}(D_1G + D_2F)\right] \qquad (50) \\
G &= A_3^TN^{-1}CN^{-1}A_3 \qquad (51)
\end{aligned}
$$

The final task is to solve this set of complicated matrix equations for the unknown matrices X, F and G. One may simplify these equations and still obtain an asymptotically correct theory by dropping higher-order terms, yielding

$$A = X - FG^{-1}F^T = N \qquad (52)$$

$$
\begin{aligned}
P &= QG^{-1}(F^TN^{-1}A_3 + D_1G + D_2F) - (FG^{-1}D_2 - D_3)^TN^{-1}CN^{-1}A_3 \\
&\quad - A^{-1}BN^{-1}A_3 - A^{-1}DN^{-1}\left[(FG^{-1}D_2 - D_3)A_3 - A_3G^{-1}(D_1G + D_2F)\right] \qquad (53)
\end{aligned}
$$

$$A_3^TN^{-1}CN^{-1}A_3 = G \qquad (54)$$

All the difficulty of solving these equations comes from the D_i matrices. Note that these matrices contain the initial curvature and twist parameters, which are characterized as small in our problem. Thus, one can solve these equations by a perturbation method. We perturb the unknowns symbolically yielding

$$
\begin{aligned}
F &= F_0 + kF_1 + k^2F_2 \\
G &= G_0 + kG_1 + k^2G_2 \qquad (55)
\end{aligned}
$$

where k is a scalar with a magnitude characteristic to that of our initial curvatures. We are not interested in finding F_2 and G_2 because these two terms will not make any contribution to a second-order asymptotically correct energy. From Eq. (52) we know, as soon as we have found G_0, G_1, F_0 and F_1, we can find X. The zeroth perturbations of Eqs. (53) and (54) are

$$-AQG_0^{-1}F_0^TN^{-1}A_{30} + BN^{-1}A_{30} = 0 \qquad (56)$$

$$A_{30}^T N^{-1} C N^{-1} A_{30} = G_0 \tag{57}$$

where

$$A_{30} = -QG_0 \tag{58}$$

One can obtain the zeroth-order equation from Eqs. (56) and (57) as

$$(N^{-1}Q)^T C N^{-1} Q = G_0^{-1} \tag{59}$$

$$F_0 = B^T A^{-1} Q G_0 \tag{60}$$

Then from Eq. (52) one obtains X.

The first-order equations from Eqs. (53) and (54) are

$$A_{30}^T N^{-1} C N^{-1} A_{31} + A_{31}^T N^{-1} C N^{-1} A_{30} = G_1 \tag{61}$$

$$\begin{aligned} &-QG_0^{-1}(F_1^T - D_1 Q^T A - D_2 B^T - G_0^{-1} G_1 F_0^T) + (F_0 G_0^{-1} D_2 - D_3)^T N^{-1} C \\ &= P G_0^{-1} Q^T N - A^{-1} D N^{-1} \left[(F_0 G_0^{-1} D_2 - D_3) Q Q^T N - Q(D_1 Q^T N + D_2 B^T) \right] \end{aligned} \tag{62}$$

where $A_{31} = \hat{A}_{31} - QG_1$ and $\hat{A}_{31}$ is a known quantity which can be calculated from F_0 and G_0. One can then obtain G_1 and F_1 and then can finally obtain X. The special case of a prismatic beam has equations that correspond to the zeroth-order equations, in which case the stiffness matrices are solved exactly; that is, the A, B, and C matrices are not corrected by initial curvatures and twist.

Numerical Results

All the above theory has been implemented into the engineering software *VABS*. In this section, numerical results from this code for both isotropic and anisotropic cases are presented and compared with available published results.

Prismatic Beams

Popescu and Hodges (1999b) presents several sets of numerical results for various prismatic beams to validate the theory introduced there. The present theory reproduces almost all the results there, even though the two approaches are very different from several perspectives. The reason why the results agree with each other to such a great extent is primarily due to both approaches being based on asymptotic methods. Another reason is that both of these methodologies use the equilibrium equations to get rid of the derivatives of strains in the asymptotically correct energy. It should be noted that the slight inconsistency invoked in the previous approach can be considered to be an approximation.

Strictly speaking, our results must be validated against full 3-D finite element analyses and experiments. This aspect of the work is still preliminary, but what has been done to date is reported in Yu *et al.* (2001). Another excellent source of validation is against *NABSA*, an existing beam sectional analysis of sufficient generality to offer meaningful comparisons. Although *NABSA* is not based on asymptotic considerations and thus cannot be considered to be the standard of our validation, one should expect our results to agree reasonably well with it. Indeed, we have found excellent agreement with results obtained from *NABSA* for many cases, including both isotropic and anisotropic beams. While there were only minor differences in most cases with *NABSA* and the analysis of Popescu and Hodges (1999b), there were some cases in which the agreement, particularly with the shear stiffnesses, was surprisingly poor. Thus, it is somewhat heartening that we have cases in which the present results agree better with *NABSA* than those of Popescu and Hodges (1999b) did.

For example, we consider a box-beam with the material properties as specified in Table 1. This case has an almost fully populated stiffness matrix. For *NABSA* we have a 16×6 mesh along the width and a 12×6 along the height. The total mesh has 336 9-noded quadralateral elements having 4368 degrees of freedom in total. For *VABS* we have a 15×6 mesh along the width and 10×6 along the height. The total mesh has 300 6-noded quadralateral elements having 2100 degrees of freedom in all. The present results agree very well with *NABSA*, as shown in Table 2. The 6×6 stiffness matrix is arranged as: 1-extension; 2, 3-shear; 4-torsion; 5, 6-bending, and the units associated with stiffness values are S_{ij} (lb.), $S_{i,j+3}$ (lb. in.), and $S_{i+3,j+3}$ (lb. in.^{2}) for $i,\ j = 1,\ 2,\ 3$. Here $VABS_R$ is the inverse of the 4×4 subset of 6×6 flexibility matrix corresponding to classical strain measures. It is also noted that the fact this layup configuration exhibits strong bending-torsion coupling, which is even larger than the torsional stiffness, can be used for the benefit of elastically tailoring airplane wings to meet requirements of aeroelastic stability.

Curved/twisted beams

So that readers will understand some of the motivation for the results presented herein, some explanation is needed. For initially curved or twisted beams, there are not many published results with which to compare. Indeed, there are only two papers on this topic known to the authors, Berdichevsky and Starosel'skii (1983) and Borri *et al.* (1992). As mentioned previously, Berdichevsky and Starosel'skii (1983) used a change of variables to construct the strain energy including transverse shear deformation. The resulting transverse shear measures combine information from the original transverse shear measures and the derivatives of the bending measures. The clear kinematical meaning of these strain measures is lost in this process, and they are obviously different from our definition here. Thus, as shown by Yu *et al.* (2001), the shear stiffness coefficients for isotropic beams derived from Berdichevsky and Starosel'skii (1983) do not agree with those of the elasticity solution. However, this does not mean that we cannot extract some information from this work that is useful for comparison. For example, it has been verified that the asymptotically correct strain energy up to the second order expressed in Eq. (32) is the same as that which is obtained in Berdichevsky and Starosel'skii (1983) for beams made with isotropic materials. Also, all the stiffnesses related only to extension and torsion should agree because these terms are not affected by the change of variables chosen therein. As for the other reference, the 1-D strain measures for the beam theory of Borri *et al.* (1992) are of the traditional kinematical meaning and thus stiffnesses could be compared were they available. Unfortunately for us, most of the results reported in that work are contour plots of 3-D stresses and thus not in a form that facilitates quantitative comparisons. It should be noted that validation of the present theory with 3-D FEM has been dealt with extensively by Yu *et al.* (2001) and will not be repeated here.

To illustrate how initial curvatures and twist affect the 6×6 stiffness matrix, we first investigate an isotropic beam with square cross section. The cross section has dimensions 0.5 in.×0.5 in. meshed with 8×8 8-noded quadrilateral elements (a total of 675 degrees of freedom), and the material properties are taken to be $E = 2.6 \times 10^7$ psi and $\nu = 0.3$. The results of *VABS* and those kindly provided by Mantegazza (2001), based on the code of Borri *et al.* (1992), are shown in Tables 3, 4 and 5. The prismatic results agree with known exact solutions very well. It can be concluded from Table 3 that when there is nonzero initial twist, there will be shear-bending coupling, S_{25} and S_{36}, and there will also be extension-torsion coupling S_{14}. These values are proportional to the magnitude of k_1. These observations agree with the result of Berdichevsky and Starosel'skii (1983) that the extension-twist coupling for naturally twisted beams should be $[S_{55} + S_{66} - 2(1+\nu)S_{44}]k_1$.

For a beam with initial curvature $k_2 \neq 0$, one can observe from Table 4 that there will be shear-torsion coupling S_{24} and extension-bending coupling S_{15}. These values are proportional to the magnitude of k_2.

The exact solution for an initially curved beam subjected to bending in the curvature plane can be found in Timoshenko and Goodier (1970). Although in Timoshenko and Goodier (1970) only a deep beam is treated and done so as a plane-stress problem, one could reasonably approximate the exact solution for beams with a square cross section were no other loads added. Here we consider such a beam with curvature $k_2 = 0.5$ and loaded with a pure bending moment $M_2 = 1$. In Fig. 3 is shown a comparison of results for this beam produced by *VABS* with this exact solution for the axial stress in the x_3 direction at $x_2 = 0$.

The solid line is the exact solution, the symbols are the results produced by *VABS*, and the dashed line is the result for the straight bar. Fig. 4 compares results for the radial stress, which is zero for a straight bar. The fact that *VABS* can yield such great accuracy in comparison to the exact solution with such a coarse mesh (8×8 parabolic elements) exemplifies the efficacy of our beam model to reproduce 3-D results.

If the beam is initially twisted (nonzero k_1) and curved (nonzero k_3), we will get other effects beside those obtained for each case separately as in Tables 3 and 4. From Table 5, we will also get a nonzero value of extension-shear coupling due to the combination of initial curvature and twist that is proportional to $k_1 + k_3$.

It is significant to note that *VABS* results correlate very well with those of Borri *et al.* (1992), a quite different approach from the present work. The only discrepancy pertains to the very small shear-torsion couplings which, although they have similar absolute values, possess different signs. The reason of this discrepancy is unknown.

To exemplify the effects of initial curvature and twist on the stiffnesses of composite beams, a CUS box-beam made of the same material as in Table 1 with $\nu_{lt} = 0.3$ and $\nu_{tn} = 0.34$ is considered. The layup in every wall is $[15°]_6$, and the mesh is the same as for the previous box-beam. The results are presented in Table 6. (The prismatic result calculated by *NABSA* is listed for use below.) This box-beam exhibits extension-twist (S_{14}) and shear-bending (S_{25} and S_{26}) couplings, even if it is prismatic. When the beam becomes initially twisted, the values of the stiffnesses are affected significantly. However, no new couplings are introduced. Indeed, we already know that initial twist k_1 only introduces bending-shear and extension-twist couplings, which already exist in the stiffness matrix of the same beam when it is regarded as prismatic. However, when the same composite box-beam is initially curved, the stiffness model will be significantly modified from the prismatic one. For the purpose of demonstrating that our theory and the associated code have the ability to model such structures, the result for the composite box-beam with initial curvature is shown in Table 7.

Locating Shear Center

For the isotropic case the shear center is defined as the point through which a transverse force will only cause transverse displacement without twist, Fung (1993). This definition is not appropriate for composite beams. Instead, it is the point on the cross section for which an applied transverse force directly induces no twist, although bending caused by the transverse force could induce twist because of bending-twist coupling. The locus of shear centers for cross sections along the beam axis is called the elastic axis of the beam. The elastic axis is a natural reference line in describing elastic deformation of the beam since it can lead to simpler resulting governing equations. To locate this axis, one has to find the shear center as accurately as possible, especially for thin-walled beams with open cross sections. There the torsional stiffness is much smaller than the bending stiffness, and even a very small error in locating the shear center may result in undesirable twisting, with respect to which the design has to be made robust. Using the thin-walled assumption, it is not difficult to locate the shear center for homogeneous beams made from isotropic materials. But for thick-walled or composite beams, it is very difficult or even impossible to find a closed-form solution for such a point. However, if one has an accurate 6×6 stiffness matrix, finding the shear center becomes trivial. Suppose one puts the beam constitutive law into the form

$$\begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} \tag{63}$$

To obtain the shear center, it is sufficient to assume that there are two transverse forces acting on a cross section, $\hat{F}_2$ and $\hat{F}_3$ (see Fig. 5). The twisting moment introduced by the transverse forces is thus

$$M_1 = \hat{F}_3 e_2 - \hat{F}_2 e_3 \tag{64}$$

where e_2 and e_3 are Cartesian coordinates of the shear center from the reference line. We need to find e_2 and e_3 to locate a position where an application of the transverse forces on the cross section results in twist $\kappa_1 = 0$. This can be written in terms of loading and stiffness as:

$$(S_{24} - S_{44}e_3)\hat{F}_2 + (S_{34} + S_{44}e_2)\hat{F}_3 = 0 \tag{65}$$

Since this equation is valid for any arbitrary $\hat{F}_2$ and $\hat{F}_3$, the location of shear center can be easily obtained as

$$\begin{aligned} e_2 &= -\frac{S_{34}}{S_{44}} \\ e_3 &= \frac{S_{24}}{S_{44}} \end{aligned} \tag{66}$$

As the first example, we investigate the shear center of a channel section (see Fig. 6). Using the thin-walled assumption, the elasticity solution for the shear center turns out to be

$$\frac{e}{b} = \frac{1}{3} - \frac{1}{12}\left(\frac{h}{b}\right)^2 + \frac{1}{\frac{5}{3}\left(\frac{h}{b}\right)^2 + \frac{7}{3}} \tag{67}$$

A plot of the shear location normalized by the thin-walled solution for various values of the ratio b/h is shown in Fig. 7. The mesh varies depending of the aspect ratio. For $b/h = 10$, the channel section is meshed by 472 8-noded quadralateral elements for a total of 4983 degrees of freedom. It is obvious that when the aspect ratio is very large, the shear center location calculated from the *VABS* 6×6 stiffness matrix converges to the result of thin-walled theory.

Effect of Stiffness Model on 1-D Results

If one can find a one-to-one correspondence between the resulting stiffness matrices for different models, it can be confidently claimed that they are equivalent to each other as far as the cross-sectional constants concerned. However, only looking at the stiffness matrix will not provide much insight. Moreover, sometimes comparing only the values of beam stiffness matrix can be very deceiving if two models derived from different methodologies. One such example is the problem considered in Table 6. The true basis for comparison should be to find how two different models maintain the strain energy in the simplified 1-D model as close to 3-D energy as possible. There are at least two ways to carry out this comparison. One is to extract the 1-D information from more complex models (2-D or 3-D models) based on the definition of 1-D variables of the beam models. For the seemingly quite different stiffness matrix in Table 6, one natural way to assess the difference is to compare a 1-D result (like a tip defection under a transverse tip force) using various stiffness models with results from complex model. Here an *ABAQUS* shell model is used. The 1-D information is obtained by using the 1-D generalized stiffness model as input to a finite element code based on the geometrically exact mixed formulation of beam theory presented in Hodges (1990a). The plots are shown in Fig. 9. The dashed line shows results from the *NABSA* model and the solid line shows the results from the *VABS* model. The mesh of the cross section is the same as that for the previous studied box-beam of Table 1. The symbols depict results from *ABAQUS*. Fig. 8 shows the undeformed shape and deformed shape finite element model using 2016 8-noded shell elements of the type S8R type with a total of 36,960 degrees of freedom when the aspect ratio L/a is approximately 5. These three sets of results are very close. The *VABS* curve appears to be closer to the *ABAQUS* results when the beam is slender, and the *NABSA* results do a bit better in the limit of the beam becoming fat (*i.e.*, ceasing to be a beam). However, it should be noted that the saving of computational effort of *VABS* and *NABSA* compared to *ABAQUS* is orders of magnitude. Moreover, a detailed 3-D analysis for this structure when L/a is large where a is the width of the box-beam and L is the length of the beam using *ABAQUS* easily overpowers an advanced workstation. In any case, it is clear that both models capture the essential behavior and that there is very little difference among the 1-D results predicted by *VABS*, *NABSA* and *ABAQUS*.

Concluding Remarks

A generalized cross-sectional modeling approach for anisotropic beams has been presented in this paper. A 6×6 cross-sectional stiffness matrix is obtained along with the interior solution of the warping for initially twisted and curved composite beams. This analysis is based on the powerful mathematical tool, the *Variational-Asymptotic Method*, which is used to asymptotically decouple the originally complex 3-D elasticity problem into a (typically) linear 2-D cross sectional analysis and a nonlinear 1-D beam problem. The ambition of helicopter researchers to develop an efficient, general and yet reliable composite beam model that can be readily incorporated in a comprehensive rotorcraft analysis has been achieved. The engineering software *VABS* has been extended to include the present theory and is able to take into consideration anisotropic, nonhomogeneous materials and to represent general cross-sectional geometries, requiring neither the costly use of 3-D finite element discretization nor the loss of accuracy inherent in simplified representations of the cross section. It is also possible to use the formulation herein to recover the 3-D stress and strain fields. (Work on validating 3-D results by comparison with 3-D finite element results and experimental data has been reported by Yu *et al.* (2001), and additional results will be reported in future papers.) The main contributions of the present paper are as follows:

1. It has been shown that for initially curved and twisted composite beams, a Timoshenko-like model cannot in general be an asymptotically correct theory. The best one can do is make certain compromises either in the form of the model or in the asymptotical correctness.

2. Given an asymptotically correct theory in terms of the classical beam 1-D strain measures and their spatial derivatives, there is a corresponding and unique Timoshenko-like model. This model can be obtained without the optimization procedure reported in previous work, Popescu and Hodges (1999b), and without the use of the so-called intrinsic warping constraint, Borri and Merlini (1986).

3. The means to capture all the elastic couplings caused by initial curvature, twist and anisotropic materials effects has been developed. Such couplings as extension- and bending-shear couplings (see Tables 3 and 5) will no doubt have influence on static and dynamic behavior of initially curved and twisted composite beams that either are short or are vibrating in modes with wavelengths that are short relative to the beam length, where in either case one still must have $h^3 << \ell^3$.

References

Berdichevsky, V. L. (1976). Equations of the theory of anisotropic inhomogeneous rods. *Dokl. Akad. Nauk SSR* 228, 558 – 561.

Berdichevsky, V. L. and Kvashnina, S. S. (1976). On equations describing the transverse vibrations of elastic bars. *PMM* 40, 120 – 135.

Berdichevsky, V. L. and Starosel'skii, L. A. (1983). On the theory of curvilinear Timoshenko-type rods. *Prikl. Matem. Mekhan.* 47, 809 – 817.

Borri, M., Ghiringhelli, G. L. and Merlini, T. (1992). Linear analysis of naturally curved and twisted anisotropic beams. *Composites Engineering* 2, 433 – 456.

Borri, M. and Merlini, T. (1986). A large displacement formulation for anisotropic beam analysis. *Meccanica* 21, 30 – 37.

Cesnik, C. E. S. and Hodges, D. H. (1997). VABS: a new concept for composite rotor blade cross-sectional modeling. *Journal of the American Helicopter Society* 42, 27 – 38.

Cesnik, C. E. S., Hodges, D. H. and Sutyrin, V. G. (1996). Cross-sectional analysis of composite beams including large initial twist and curvature effects. *AIAA Journal* 34, 1913 – 1920.

Danielson, D. A. and Hodges, D. H. (1987). Nonlinear beam kinematics by decomposition of the rotation tensor. *Journal of Applied Mechanics* 54, 258 – 262.

Fung, Y. C. (1993). An Introduction to the Theory of Aeroelasticity. Dover, New York, USA, 1st edition.

Giavotto, V., Borri, M., Mantegazza, P., Ghiringhelli, G., Carmaschi, V., Maffioli, G. C. and Mussi, F. (1983). Anisotropic beam theory and applications. *Computers and Structures* 16, 403 – 413.

Hodges, D. H. (1990a). A mixed variational formulation based on exact intrinsic equations for dynamics of moving beams. *International Journal of Solids and Structures* 26, 1253 – 1273.

Hodges, D. H. (1990b). A review of composite rotor blade modeling. *AIAA Journal* 28, 561 – 565.

Hodges, D. H., Atilgan, A. R., Cesnik, C. E. S. and Fulton, M. V. (1992). On a simplified strain energy function for geometrically nonlinear behaviour of anisotropic beams. *Composites Engineering* 2, 513 – 526.

Jung, S. N., Nagaraj, V. T. and Chopra, I. (1999). Assessment of composite rotor blade modeling techniques. *Journal of the American Helicopter Society* 44, 188 – 205.

Mantegazza, P. (2001). Private communication.

Popescu, B. and Hodges, D. H. (1999a). Asymptotic treatment of the trapeze effect in finite element cross-sectional analysis of composite beams. *International Journal of Non-Linear Mechanics* 34, 709–721.

Popescu, B. and Hodges, D. H. (1999b). On asymptotically correct timoshenko-like anisotropic beam theory. *International Journal of Solids and Structures* 37, 535–558.

Popescu, B., Hodges, D. H. and Cesnik, C. E. S. (2000). Obliqueness effects in asymptotic cross-sectional analysis of composite beams. *Computers and Structures* 76, 533 – 543.

Sutyrin, V. G. (1997). Derivation of plate theory accounting asymptotically correct shear deformation. *Journal of Applied Mechanics* 64, 905 – 915.

Timoshenko, S. P. and Goodier, J. N. (1970). Theory of Elasticity. McGraw-Hill, Maidenhead, U.K.

Volovoi, V. V., Hodges, D. H., Cesnik, C. E. S. and Popescu, B. (1999). Assessment of beam modeling methods for rotor blade applications. In Proceedings of the 55th Annual Forum of the American Helicopter Society, Montréal, Quebec, Canada.

Yu, W., Volovoi, V. V. and Hodges, D. H. (2001). Validation of the variational asymptotic beam sectional (VABS) analysis. In Proceedings of the 42nd Structures, Structural Dynamics and Materials Conference, Seattle, Washington. AIAA Paper 2001-1530.

List of Table Captions

outer dimensions:	width $a = 0.953$ in
thickness: $h = 0.03$	height $b = 0.53$ in
CAS1 – right wall:	$[15°/-15°]_3$
left wall:	$[-15°/15°]_3$
upper wall:	$[-15°]_6$
lower wall:	$[15°]_6$
material properties:	$E_l = 20.59 \times 10^6$ psi
$E_t = 1.42 \times 10^6$ psi	$G_{lt} = 8.7 \times 10^5$ psi
$G_{tn} = 6.96 \times 10^5$ psi	$\nu_{lt} = \nu_{tn} = 0.42$

Table 1:

S	*NABSA*	Popescu *et al.*	Present	$VABS_R$
S_{11}	0.137×10^7	0.137×10^7	0.137×10^7	0.985×10^6
S_{12}	-0.184×10^6	-0.184×10^6	-0.184×10^6	–
S_{13}	-0.150×10^3	0.176×10^4	-0.131×10^3	–
S_{22}	0.885×10^5	0.883×10^5	0.884×10^5	–
S_{23}	0.803×10^2	-0.842×10^3	0.719×10^2	–
S_{33}	0.387×10^5	0.775×10^4	0.389×10^5	–
S_{44}	0.170×10^5	0.174×10^5	0.170×10^5	0.170×10^5
S_{45}	0.176×10^5	0.180×10^5	0.176×10^5	0.176×10^5
S_{46}	-0.349×10^3	-0.362×10^3	-0.351×10^3	-0.351×10^3
S_{55}	0.591×10^5	0.608×10^5	0.591×10^5	0.591×10^5
S_{56}	-0.371×10^3	-0.372×10^3	-0.371×10^3	-0.372×10^3
S_{66}	0.141×10^6	0.143×10^6	0.141×10^6	0.141×10^6

TABLE 2:

S	Prismatic	k_1=0.05		k_1=0.10	
		Borri *et al.* (1992)	*VABS*	Borri *et al.* (1992)	*VABS*
S_{11}	0.650×10^7	0.650×10^7	0.650×10^7	0.650×10^7	0.650×10^7
S_{14}	0	0.212×10^4	0.212×10^4	0.423×10^4	0.424×10^4
S_{22}	0.207×10^7	0.207×10^7	0.207×10^7	0.207×10^7	0.207×10^7
S_{25}	0	-0.136×10^4	-0.136×10^4	-0.271×10^4	-0.271×10^4
S_{33}	0.207×10^7	0.207×10^7	0.207×10^7	0.207×10^7	0.207×10^7
S_{36}	0	-0.136×10^4	-0.136×10^4	-0.271×10^4	-0.271×10^4
S_{44}	0.879×10^5	0.879×10^5	0.879×10^5	0.879×10^5	0.879×10^5
S_{55}	0.135×10^6	0.135×10^6	0.135×10^6	0.135×10^6	0.135×10^6
S_{66}	0.135×10^6	0.135×10^6	0.135×10^6	0.135×10^6	0.135×10^6

Table 3:

S	Prismatic	k_2=0.05		k_2=0.10	
		Borri *et al.* (1992)	*VABS*	Borri *et al.* (1992)	*VABS*
S_{11}	0.650×10^7	0.650×10^7	0.650×10^7	0.650×10^7	0.650×10^7
S_{15}	0	-0.934×10^4	-0.880×10^4	-0.187×10^5	-0.176×10^5
S_{22}	0.207×10^7	0.207×10^7	0.207×10^7	0.207×10^7	0.207×10^7
S_{24}	0	0.218×10^4	-0.226×10^4	0.436×10^4	-0.452×10^4
S_{33}	0.207×10^7	0.207×10^7	0.207×10^7	0.207×10^7	0.207×10^7
S_{44}	0.879×10^5	0.879×10^5	0.879×10^5	0.879×10^5	0.879×10^5
S_{55}	0.135×10^6	0.135×10^6	0.135×10^6	0.135×10^6	0.135×10^6
S_{66}	0.135×10^6	0.135×10^6	0.135×10^6	0.135×10^6	0.135×10^6

TABLE 4:

S	Prismatic	k_1=0.05, k_3=0.05		k_1=0.10, k_3=0.10	
		Borri *et al.* (1992)	*VABS*	Borri *et al.* (1992)	*VABS*
S_{11}	0.650×10^7	0.650×10^7	0.650×10^7	0.650×10^7	0.650×10^7
S_{13}	0	0.253×10^3	0.253×10^3	0.101×10^4	0.101×10^4
S_{14}	0	0.212×10^4	0.212×10^4	0.424×10^4	0.424×10^4
S_{16}	0	-0.934×10^4	-0.880×10^4	-0.187×10^5	-0.176×10^5
S_{22}	0.207×10^7	0.207×10^7	0.207×10^7	0.207×10^7	0.207×10^7
S_{25}	0	-0.136×10^4	-0.136×10^4	-0.271×10^4	-0.271×10^4
S_{33}	0.207×10^7	0.207×10^7	0.207×10^7	0.207×10^7	0.207×10^7
S_{34}	0	0.218×10^4	-0.226×10^4	0.436×10^4	-0.451×10^4
S_{36}	0	-0.136×10^4	-0.136×10^4	-0.272×10^4	-0.272×10^4
S_{44}	0.879×10^5	0.879×10^5	0.879×10^5	0.879×10^5	0.879×10^5
S_{55}	0.135×10^6	0.135×10^6	0.135×10^6	0.135×10^6	0.135×10^6
S_{66}	0.135×10^6	0.135×10^6	0.135×10^6	0.135×10^6	0.135×10^6

TABLE 5:

S	*NABSA*	*VABS*	k_1=0.05	k_1=0.10
S_{11}	0.1437×10^7	0.1437×10^7	0.1317×10^7	0.9453×10^6
S_{14}	0.1074×10^6	0.1074×10^6	0.9302×10^5	0.4036×10^5
S_{22}	0.9024×10^5	0.5042×10^5	0.5421×10^5	0.5750×10^5
S_{25}	-0.5200×10^5	-0.2905×10^5	-0.2948×10^5	-0.2931×10^5
S_{33}	0.3940×10^5	0.2058×10^5	0.2515×10^5	0.3036×10^5
S_{36}	-0.5635×10^5	-0.2941×10^5	-0.3115×10^5	-0.3300×10^5
S_{44}	0.1679×10^5	0.1679×10^5	0.1451×10^5	0.6347×10^4
S_{55}	0.6621×10^5	0.5298×10^5	0.545×10^5	0.5564×10^5
S_{66}	0.1725×10^6	0.1340×10^6	0.1383×10^6	0.1436×10^6

Table 6:

S	Prismatic	k_2=0.01	k_2=0.05
S_{11}	0.1437×10^7	0.1486×10^7	0.2060×10^7
S_{12}	0.0	-0.4884×10^5	-0.1248×10^6
S_{14}	0.1074×10^6	0.1150×10^6	0.2052×10^6
S_{15}	0.0	0.2721×10^5	0.9590×10^5
S_{22}	0.5042×10^5	0.4924×10^5	0.2500×10^5
S_{24}	0.0	-0.7661×10^4	-0.1960×10^5
S_{25}	-0.2905×10^5	-0.2869×10^5	-0.2020×10^5
S_{33}	0.2058×10^5	0.19914×10^5	0.7436×10^4
S_{36}	-0.2941×10^5	-0.28827×10^5	-0.1656×10^5
S_{44}	0.1679×10^5	0.1799×10^5	0.3216×10^5
S_{45}	0.0	0.4331×10^4	0.1527×10^5
S_{55}	0.5298×10^5	0.5240×10^5	0.3845×10^5
S_{66}	0.1340×10^6	0.1326×10^6	0.1018×10^6

TABLE 7:

List of Figure Captions

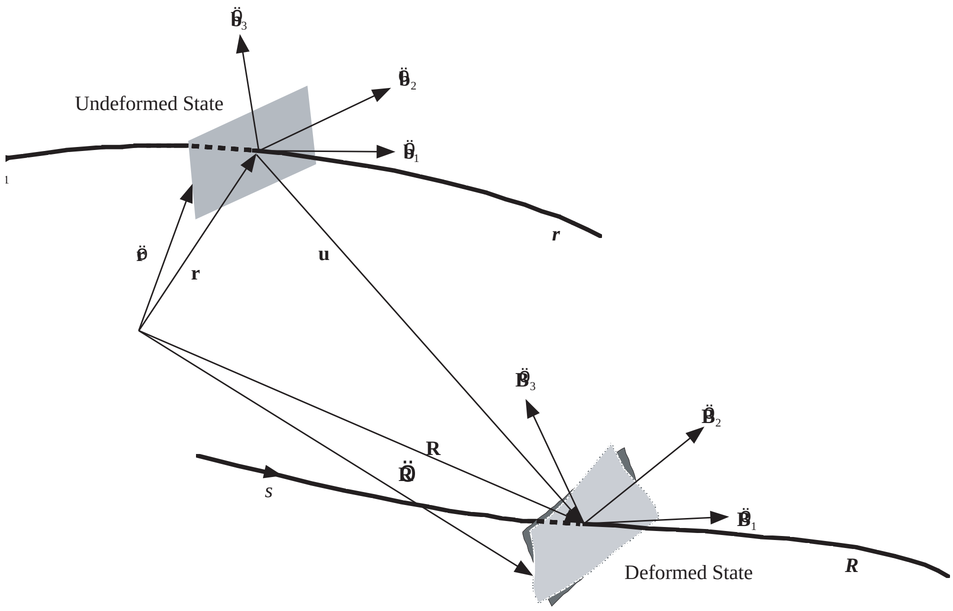

FIGURE 1:

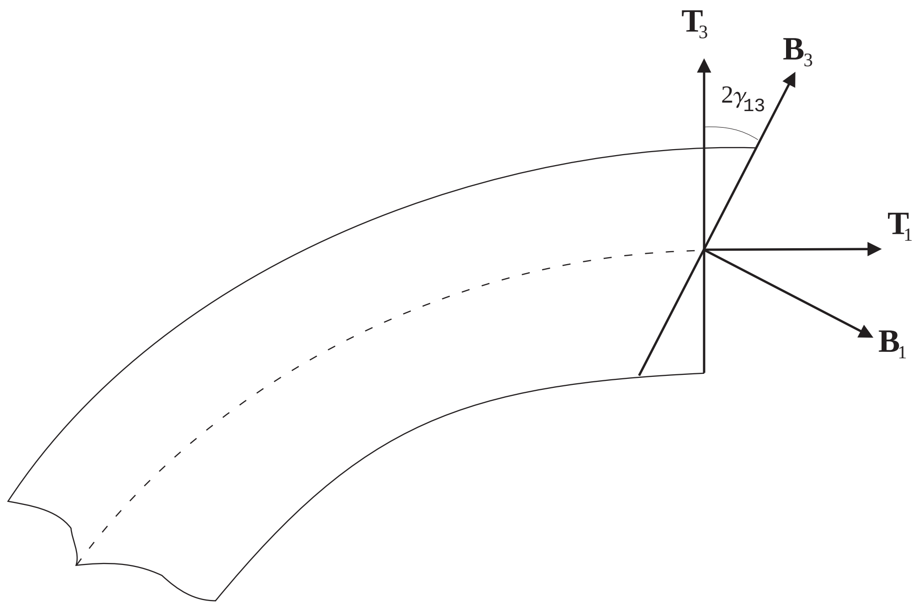

FIGURE 2:

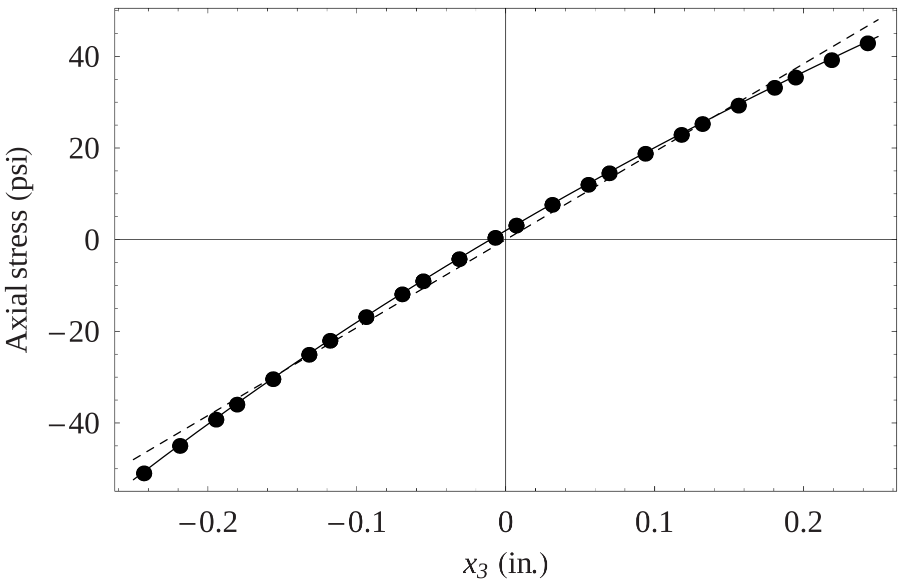

FIGURE 3:

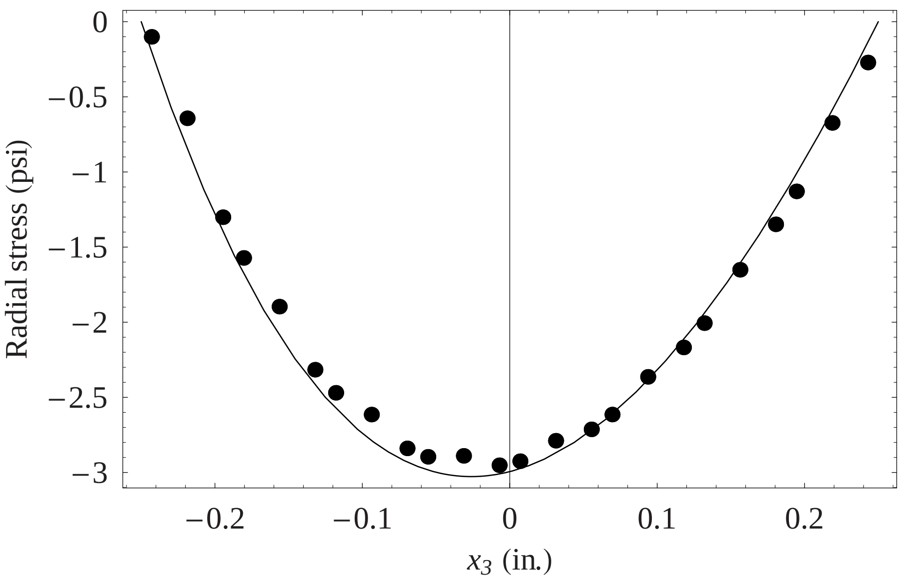

FIGURE 4:

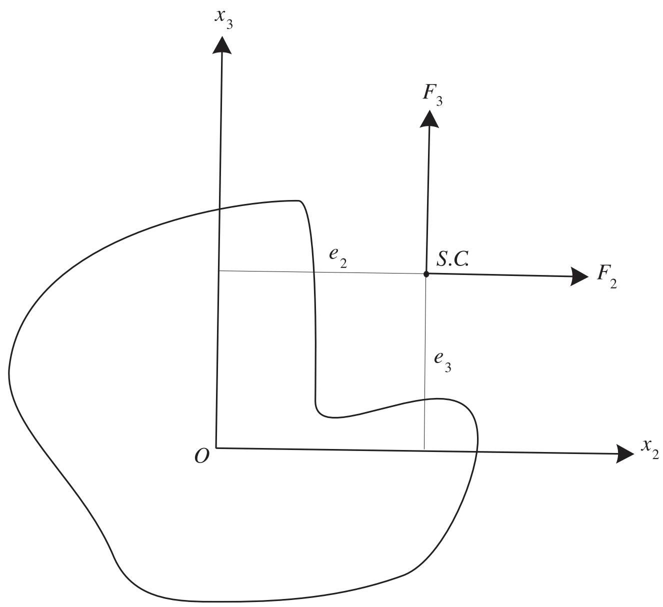

FIGURE 5:

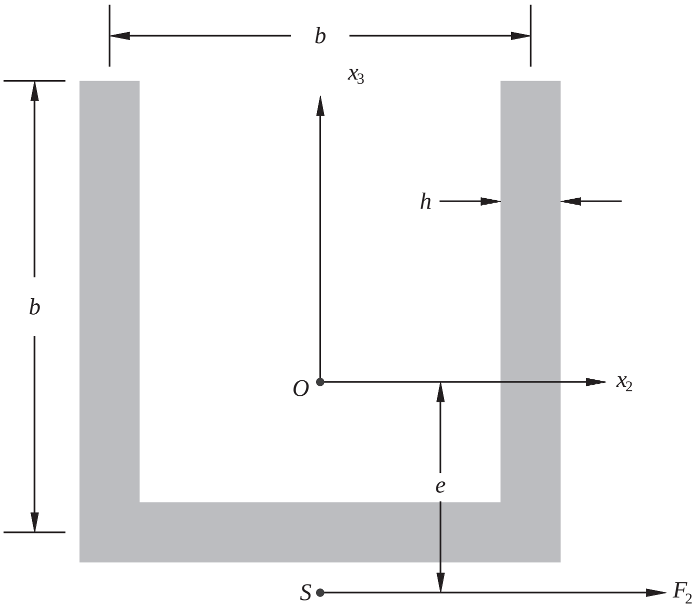

Figure 6:

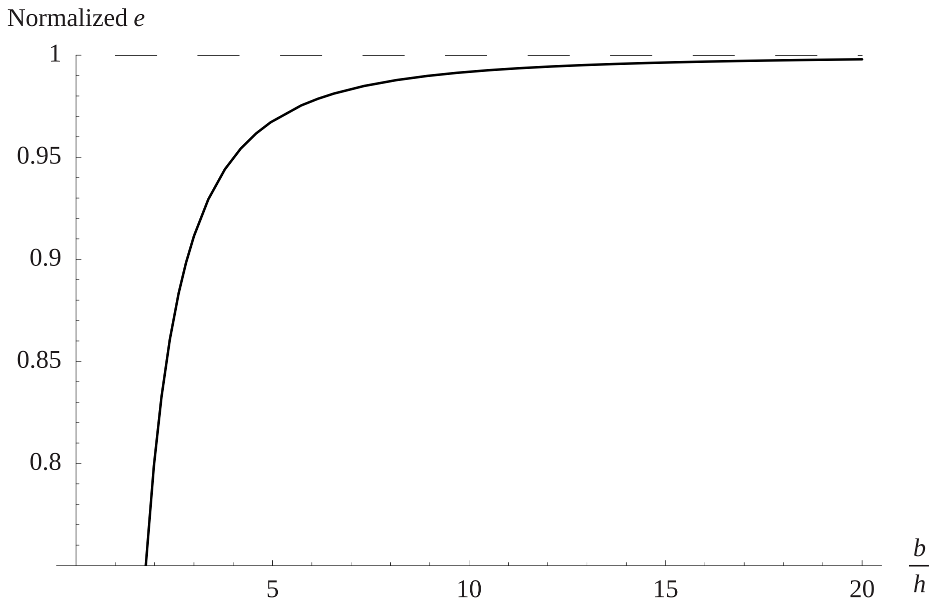

Figure 7:

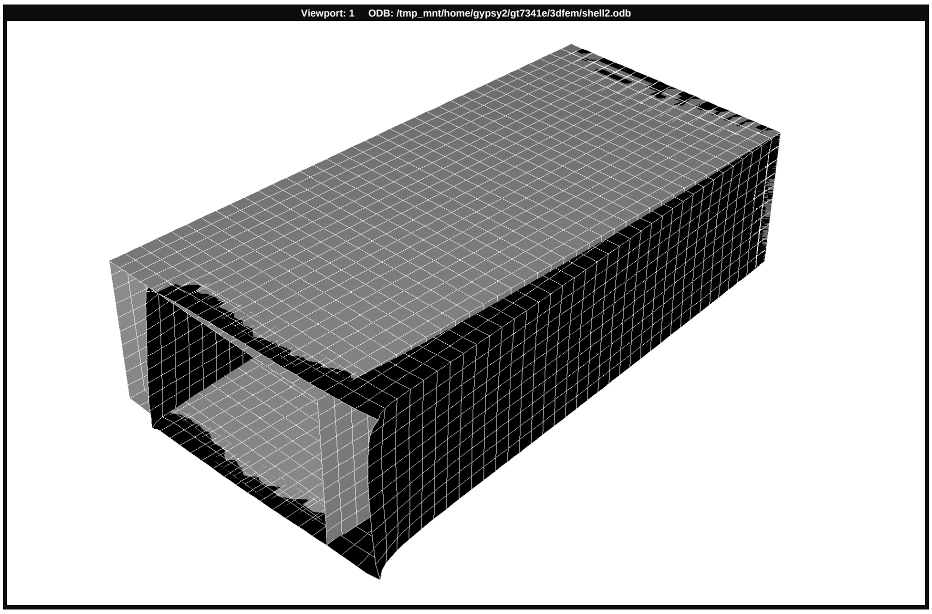

FIGURE 8:

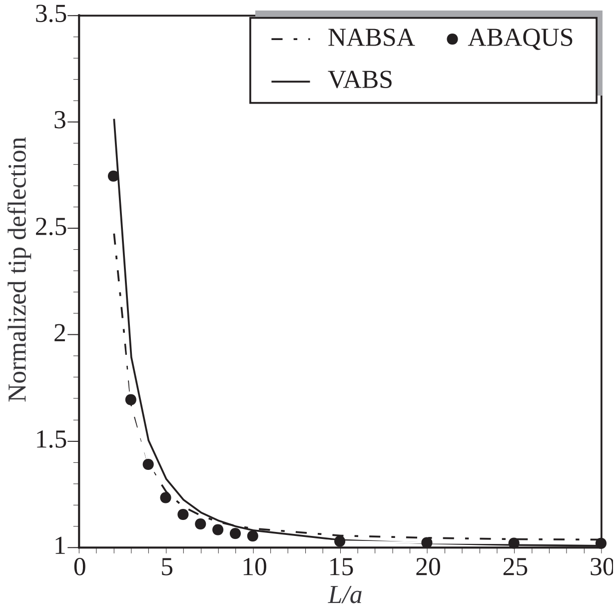
NABSA
ABAQUS
VABS
Normalized tip deflection
L/a
3.5
3
2.5
2
1.5
1
0
5
10
15
20
25
30

FIGURE 9: