

# Theory of Initially Twisted, Composite, Thin-walled Beams

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## Abstract

An asymptotically correct theory for initially-twisted, thin-walled, composite beams has been constructed by the variational asymptotic method. The strain energy of the original, three-dimensional structure is first rigorously reduced to be a two-dimensional energy expressed in terms of shell strains. Then the two-dimensional strain energy is further reduced to be expressed in terms of the classical beam strain measures. The resulting theory is a classical beam model approximating the three-dimensional energy through the first order of the initial twist. Consistent use of small parameters that are intrinsic to the problem allows a natural derivation for all thin-walled beams within a common framework, regardless of whether the section is open, closed, or strip-like. Several examples are studied using the present theory and the results are compared with a general cross-sectional analysis, VABS, and other published results.

**Keywords:** composite beams, thin-walled, initial twist, asymptotic method, VABS

## Introduction

A thin-walled beam is characterized as a flexible body that has different magnitudes for all three of its characteristic dimensions [1]. To be classified as a beam,  $c$ , the characteristic dimension of the cross section, must be much smaller than  $l$ , the wavelength of the deformation along the beam, i.e.  $c/l \ll 1$ . Moreover, for a beam to be classified as thin-walled implies that the maximum thickness of the walls,  $h$ , is much smaller than  $c$ , so

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that  $h/c \ll 1$ . Although one can analyze thin-walled beams using three-dimensional (3-D) elasticity theory, thin-walled beam theories take advantage of the small parameters,  $h/c$  and  $c/l$ , to derive a one-dimensional (1-D) model. This model consists of 1-D constitutive equations (cross-sectional elastic constants) and “recovery relations.” The former are used in the 1-D equilibrium and kinematic equations to analyze the original 3-D structure, and the latter provide approximate values of the 3-D displacement, strain, and stress from the 1-D solution.

Thin-walled beam theories strive to present closed-form expressions for cross-sectional stiffness constants and stresses (or stress flows). There are mainly two types of thin-walled beam theories. The first type can be classified as *ad hoc* models [1–5]. In these models, assumptions are invoked based on engineering intuition. These can be assumptions that the beam deforms in specific modes or that certain components of the displacement/strain/stress are negligible. Usually, these assumptions are based on experience with thin-walled beams made with isotropic materials, which can be justified by some exact solutions. However, for anisotropic materials, various modes of deformation can be coupled, and these theories might fail for some special cases which cannot be properly represented by the invoked assumptions [6]. Nevertheless, some of these models such as [4,5] can provide a good prediction for many cases, and it is straightforward to refine the model by incorporate additional deformation such as transverse shear to remedy possible errors introduced by *ad hoc* assumptions.

The second type encompasses asymptotic models [6–9]. Therein, the original 3-D elasticity equations are mathematically reduced to a 1-D model using small parameters inherent to the problem. While application of traditional asymptotic methods is possible, the authors prefer the Variational Asymptotic Method (VAM) [10]. In these models, the material anisotropy is accounted for in a consistent and systematic manner, and those deformation modes that contribute most significantly to the energy emerge naturally. In our formulation, elastic couplings among all deformations are accounted for by using the 3-D material law, which uses 21 elastic constants for anisotropic materials. However, the refined models constructed directly using the VAM are of little practical use, per se. Usually some transformation, which might detract from the asymptotical correctness, has to be carried out to convert such models into a form that is of practical use for engineers [11].

The present paper was originally planned to serve as a natural extension of the work in [9] to enhance the capability of that theory to accommodate initial twist, so that more realistic problems (such as pretwisted composite rotor blades or wind turbines) can be analyzed. It was later found out that it is very complicated, if not impossible, to incorporate the initial twist into that, already complex, formulation. Instead, the present formulation is cast in an intrinsic form and the derivation departs from previous work at the outset. First, the

3-D elasticity representation is rationally reduced to the classical shell approximation of Berdichevsky [10] with geometric correction by considering  $h/c$  as the main small parameter and taking into account all first-order corrections from the initial twist of the thin-walled beam. Then the two-dimensional (2-D) variables are expressed in terms of intrinsic beam variables and unknown warping functions. Substituting these relations back into the 2-D strain energy, which is an asymptotic approximation of the original 3-D energy, one can use the VAM to solve for the unknown warping functions to minimize the 2-D strain energy. The final result is a strain energy for the thin-walled beam with first-order correction from initial twist. For validation, several examples of thin-walled beams are studied; and the results are compared with some available in the literature and VABS[11], a general-purpose finite element program for arbitrary cross sections that does not take advantage of the smallness of the wall thickness.

## Kinematics

A general thin-walled beam can be depicted as in Fig. 1; note that this picture does not show the initial twist. Here  $O$  is a fixed point in space,  $\bar{O}$  is on the beam axis specified by the position vector  $\mathbf{r}_o$ , and  $\hat{O}$  is on the contour intersecting the reference surface (considering the thin-walled structure as a shell) with the beam section cut through the point  $\bar{O}$ . Here two dextral coordinate systems  $x_i$  and  $y_i$  are introduced;  $y_1$  is the running length coordinate along the beam axis with  $\mathbf{b}_1$  as the unit vector;  $y_a$  are the local Cartesian coordinates of the beam section with  $\mathbf{b}_a$  as the unit vectors;  $x_1$  is parallel to  $y_1$ ,  $x_2$  is the arc length along the contour, and  $x_3$  is the outward normal of the reference surface. (Here and throughout the paper, Greek indices assume values 1 and 2 while Latin indices  $i, j, \dots, z$  assume 1, 2, and 3 and  $a, b, \dots, h$  assume 2 and 3. Repeated indices are summed over their range except where explicitly indicated.)

The position vector of the shell reference surface is

$$\mathbf{r}(x_1, x_2) = \mathbf{r}_0(y_1) + y_a(x_2)\mathbf{b}_a(y_1) \quad (1)$$

The covariant base vectors of the reference surface are defined by

$$\mathbf{a}_\alpha = \mathbf{r}_{,\alpha} \quad (2)$$

where  $(\cdot)_{,\alpha} = \frac{\partial(\cdot)}{\partial x_\alpha}$ . Using Eq. (1), the covariant base vectors can be written explicitly as

$$\begin{aligned}\mathbf{a}_1 &= \mathbf{b}_1 + y_2 k_1 \mathbf{b}_3 - y_3 k_1 \mathbf{b}_2 \\ \mathbf{a}_2 &= \dot{y}_a \mathbf{b}_a \\ \mathbf{a}_3 &= \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|} = \frac{-k_1 R_\tau \mathbf{b}_1 - \dot{y}_3 \mathbf{b}_2 + \dot{y}_2 \mathbf{b}_3}{\sqrt{1 + k_1^2 R_\tau^2}}\end{aligned}\quad (3)$$

with  $R_\tau = y_a \dot{y}_a$ ,  $(\dot{\cdot}) = \frac{\partial(\cdot)}{\partial x_2}$ , and  $k_1$  as the initial twist. The first fundamental form of the surface is obtained from  $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ , such that:

$$\begin{aligned}a_{11} &= 1 + k_1^2 R^2 \\ a_{12} &= a_{21} = k_1 R_n \\ a_{22} &= 1\end{aligned}\quad (4)$$

with  $R^2 = y_a y_a$  and  $R_n = y_2 \dot{y}_3 - y_3 \dot{y}_2$ . The determinant of the first fundamental form is

$$a = 1 + k_1^2 R_\tau^2 \quad (5)$$

The second fundamental form of the surface can be calculated by the definition

$$b_{\alpha\beta} = \mathbf{r}_{,\alpha\beta} \cdot \mathbf{a}_3 = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}_3 \quad (6)$$

such that

$$\begin{aligned}b_{11} &= \frac{k_1^2 R_n}{\sqrt{a}} \\ b_{12} &= b_{21} = \frac{k_1}{\sqrt{a}} \\ b_{22} &= \frac{\dot{y}_2 \dot{y}_3 - \dot{y}_3 \dot{y}_2}{\sqrt{a}}\end{aligned}\quad (7)$$

Any point in the undeformed 3-D structure can be described as

$$\hat{\mathbf{r}}(x_1, x_2, x_3) = \mathbf{r}(x_1, x_2) + x_3 \mathbf{a}_3 \quad (8)$$

The 3-D covariant base vectors are defined using  $\mathbf{g}_i = \hat{\mathbf{r}}_{,i}$  and can be obtained as

$$\begin{aligned}\mathbf{g}_\alpha &= \mathbf{a}_\alpha - x_3 b_\alpha^\lambda \mathbf{a}_\lambda \\ \mathbf{g}_3 &= \mathbf{a}_3\end{aligned}\quad (9)$$

The 3-D metric tensor can then be calculated as

$$\begin{aligned} g_{\alpha\beta} &= a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 b_{\alpha}^{\lambda} b_{\lambda\beta} \\ g_{i3} &= \delta_{i3} \end{aligned} \quad (10)$$

where  $\delta$  is the Kronecker symbol. From Eq. (10) one can calculate the determinant of the metric tensor

$$g = [1 - 2x_3 H + x_3^2 K]^2 a \quad (11)$$

with  $H = \frac{1}{2} b_{\alpha}^{\alpha}$  as the mean curvature and  $K = \det(b_{\alpha}^{\beta})$  as the Gaussian curvature of the surface.

### Dimensional Reduction from 3-D to 2-D

The dimensional reduction from the original 3-D thin-walled structure to a 1-D beam model can be carried out in two steps due to the existence of two different small parameters  $h/c$  and  $c/l$ . Firstly, making use of  $h/c$ , one can approximate the original 3-D energy with a 2-D energy defined in the shell reference surface. Secondly, making use of  $c/l$ , one can approximate the above-found 2-D energy with 1-D energy defined along the beam axis.

After deformation, the position of any material point in the 3-D structure can be described by the vector

$$\hat{\mathbf{R}}(x_1, x_2, x_3) = \mathbf{R}(x_1, x_2) + x_3 \mathbf{A}_3(x_1, x_2) + w_i(x_1, x_2, x_3) \mathbf{A}^i(x_1, x_2) \quad (12)$$

where the  $\mathbf{A}^i$  are the contravariant base vectors of  $\mathbf{A}_i$ , which are, in turn, the base vectors of the deformed reference surface given by

$$\begin{aligned} \mathbf{A}_{\alpha} &= \mathbf{R}_{,\alpha} \\ \mathbf{A}_3 &= \frac{\mathbf{A}_1 \times \mathbf{A}_2}{|\mathbf{A}_1 \times \mathbf{A}_2|} \end{aligned} \quad (13)$$

Here we constrain  $\mathbf{A}_3$  to be normal to the deformed reference surface by including all possible deformations of the transverse normal into the 3-D warping functions  $w_i(x_1, x_2, x_3)$ . One can define  $\mathbf{R}$  as

$$\mathbf{R} = \frac{1}{h} \langle \hat{\mathbf{R}} \rangle \quad (14)$$

which implies three constraints on the warping functions, so that

$$\langle w_i(x_1, x_2, x_3) \rangle = 0 \quad (15)$$

where the angle-brackets denote the definite integral through the thickness of the shell.

From Eq. (12), one can obtain the 3-D covariant base vectors and the corresponding metric tensor according to the following formulas

$$\begin{aligned}\mathbf{G}_i &= \hat{\mathbf{R}}_{,\alpha} \\ G_{ij} &= \mathbf{G}_i \cdot \mathbf{G}_j\end{aligned}\quad (16)$$

The 3-D strain tensor is defined as

$$\Gamma_{ij} = \frac{1}{2}(G_{ij} - g_{ij}) \quad (17)$$

Making use of Eqs.(17), (16), (12), and (10), one can derive the linearized 3-D strain field as

$$\begin{aligned}\Gamma_{\alpha\beta} &= \epsilon_{\alpha\beta} + x_3\kappa_{\alpha\beta} - x_3b_{(\alpha}^\lambda\epsilon_{\lambda\beta)} - x_3^2b_{(\alpha}^\lambda\kappa_{\lambda\beta)} + w_{(\alpha;\beta)} - b_{\alpha\beta}w_3 - x_3b_{(\alpha}^\lambda w_{\lambda;\beta)} + x_3b_\alpha^\lambda b_{\lambda\beta}w_3 \\ 2\Gamma_{\alpha 3} &= w_{3,\alpha} + w_\lambda b_\alpha^\lambda + w_{\alpha,3} - x_3b_\alpha^\beta w_{\beta,3} \\ \Gamma_{33} &= w_{3,3}\end{aligned}\quad (18)$$

where the semicolon preceding an index denotes the covariant derivative with respect to the coordinate, and the parentheses in the subscripts denote the symmetrization operation [12], meaning  $a_{(\alpha\beta)} = \frac{1}{2}(a_{\alpha\beta} + a_{\beta\alpha})$ . The 2-D generalized strain measures  $\epsilon_{\alpha\beta}$  and  $\kappa_{\alpha\beta}$  are defined as [13, 14].

$$\begin{aligned}\epsilon_{\alpha\beta} &= \frac{1}{2}(A_{\alpha\beta} - a_{\alpha\beta}) \\ \kappa_{\alpha\beta} &= b_{\alpha\beta} - B_{\alpha\beta} + b_{(\alpha}^\lambda\epsilon_{\lambda\beta)}\end{aligned}\quad (19)$$

where  $A_{\alpha\beta}$  and  $B_{\alpha\beta}$  are the first and second fundamental forms of the deformed surface, respectively, and defined as

$$\begin{aligned}A_{\alpha\beta} &= \mathbf{A}_\alpha \cdot \mathbf{A}_\beta \\ B_{\alpha\beta} &= \mathbf{A}_{\alpha,\beta} \cdot \mathbf{A}_3\end{aligned}\quad (20)$$

Note that the 3-D strains, as in Eq. (18), are obtained based on the assumption that the 2-D generalized strains are small and the warping functions are of the order of the generalized strains. This means all the products between warping functions and generalized strains are neglected. Other than this, all the 3-D deformation is accounted in this strain field. In view of the smallness of  $h/c$  and initial twist  $k_1$ , the main terms of the 3-D strain field which will

contribute to the zeroth-order energy are

$$\begin{aligned}
\Gamma_{\alpha\beta}^0 &= \epsilon_{\alpha\beta} + x_3\kappa_{\alpha\beta} \\
2\Gamma_{\alpha 3}^0 &= w_{\alpha,3} \\
\Gamma_{33}^0 &= w_{3,3}
\end{aligned} \tag{21}$$

Now, the 3-D strain energy of the thin-walled structure can be expressed as

$$\begin{aligned}
J &= \frac{1}{2} \int_v \Gamma^T D \Gamma \mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 dx_1 dx_2 dx_3 \\
&= \frac{1}{2} \int_s \langle \Gamma^T D \Gamma (1 - 2x_3 H + x_3^2 K) \rangle ds
\end{aligned} \tag{22}$$

where  $v$  is the volume occupied by the 3-D body in the undeformed configuration,  $s$  is the surface stretched by the undeformed reference surface and

$$\begin{aligned}
\Gamma &= [\Gamma_e \quad 2\Gamma_s \quad \Gamma_t]^T \\
\Gamma_e &= [\Gamma_{11} \quad 2\Gamma_{12} \quad \Gamma_{22}]^T \\
2\Gamma_s &= [2\Gamma_{13} \quad 2\Gamma_{23}]^T \\
\Gamma_t &= \Gamma_{33}
\end{aligned} \tag{23}$$

The strain energy per unit area (which is the same as the strain energy for the deformation of the normal-line element) is

$$U = \frac{1}{2} \langle \Gamma^T D \Gamma (1 - 2x_3 H + x_3^2 K) \rangle \tag{24}$$

where  $D$  is the 3-D  $6 \times 6$  material matrix transformed from the material system into the general curvilinear system  $x_i$  and is generally fully populated for composite materials.  $D$  is a function of initial twist  $k_1$  which may be expanded asymptotically as

$$D = D_0 + D_1 + o(k_1\mu) \tag{25}$$

where  $D_0$  is of the order of the elastic constants  $\mu$ ,  $D_1$  is of the order  $k_1\mu$ , and  $o(k_1\mu)$  represents terms of order higher than  $k_1\mu$ . For isotropic materials,  $D$  as a fourth-order tensor can be found in a typical elasticity book, which is an isotropic tensor formed by two material parameters and the 3-D metric tensor of the undeformed configuration. What one needs to do next is the put this fourth-order tensor to a  $6 \times 6$  matrix according to the engineering notation used in Eq. (23). For an anisotropic material, we need to perform

the transformation between the material system and an orthonormal system. Usually, the orthonormal system is defined in terms of the lamina plane orientation angle and the ply angle, yielding the conventional transformation for composite materials. Then we need to transform the material properties in the aforementioned orthonormal system into the curvilinear system. It is convenient to derive such relations using general tensorial notation. If  $D^{ijkl}$  represents the components of material tensor in the curvilinear system and  $\hat{D}^{mnpq}$  represents those in the orthonormal system, we have

$$D^{ijkl} \mathbf{g}_i \mathbf{g}_j \mathbf{g}_k \mathbf{g}_l = \hat{D}^{mnpq} \mathbf{s}_m \mathbf{s}_n \mathbf{s}_p \mathbf{s}_q \quad (26)$$

with  $\mathbf{s}_i$  as the unit vectors associated with the orthonormal system. From the above equation, one can easily obtain the material properties in the curvilinear system such that:

$$D^{ijkl} = \hat{D}^{mnpq} (\mathbf{s}_m \cdot \mathbf{g}_i) (\mathbf{s}_n \cdot \mathbf{g}_j) (\mathbf{s}_p \cdot \mathbf{g}_k) (\mathbf{s}_q \cdot \mathbf{g}_l) \quad (27)$$

According to the VAM, it is sufficient to find the zeroth-order warping for the purpose of obtaining an energy asymptotically correct through the first order of  $k_1$ , which is the focus of the present work. The zeroth-order energy per unit area can be written as

$$2U_0 = \left\langle \begin{Bmatrix} \Gamma_e^0 \\ 2\Gamma_s^0 \\ \Gamma_t^0 \end{Bmatrix}^T \begin{bmatrix} D_e & D_{es} & D_{et} \\ D_{es}^T & D_s & D_{st} \\ D_{et}^T & D_{st}^T & D_t \end{bmatrix} \begin{Bmatrix} \Gamma_e^0 \\ 2\Gamma_s^0 \\ \Gamma_t^0 \end{Bmatrix} \right\rangle \quad (28)$$

where  $D_e$ ,  $D_{es}$ ,  $D_{et}$ ,  $D_s$ ,  $D_{st}$ ,  $D_t$  are the corresponding partition matrices of  $D_0$ . The leading terms of warping functions in Eq. (28) are

$$2U_0^* = \langle 2\Gamma_e^{0T} D_{es} 2\Gamma_s^0 + 2\Gamma_e^{0T} D_{et} \Gamma_t^0 + 2\Gamma_s^{0T} D_s 2\Gamma_s^0 + 2\Gamma_s^{0T} D_{st} \Gamma_t^0 + \Gamma_t^{0T} D_t \Gamma_t^0 \rangle \quad (29)$$

Minimizing Eq. (29) by following the general procedure from the calculus of variations, one can solve for  $2\Gamma_s^0$  and  $\Gamma_t^0$  as

$$\begin{aligned} 2\Gamma_s^0 &= -D_s^{*-T} D_{es}^{*T} \Gamma_e^0 \\ \Gamma_t^0 &= -D_t^{-T} D_{et}^{*T} \Gamma_e^0 \end{aligned} \quad (30)$$

with

$$\begin{aligned}
D_s^* &= D_s - D_{st}D_t^{-1}D_{st}^T \\
D_{es}^* &= D_{es} - D_{et}D_t^{-1}D_{st}^T \\
D_{et}^* &= D_{et} - D_{es}^*D_s^{*-1}D_{st}
\end{aligned} \tag{31}$$

From Eqs. (15), (21) and (30), one can solve for the zeroth-order warping field. Substituting the solved warping functions into Eq. (18), one can obtain the 3-D strains asymptotically correct through the first order of  $k_1$ , which can be symbolically written as

$$\Gamma = \Gamma^0 + \Gamma^1 \tag{32}$$

The strain energy per unit area asymptotically correct through the zeroth order of  $k_1$  can be calculated from Eq. (24) as

$$2U_0 = \begin{Bmatrix} \epsilon \\ \kappa \end{Bmatrix}^T \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^T & \mathcal{D} \end{bmatrix} \begin{Bmatrix} \epsilon \\ \kappa \end{Bmatrix} \tag{33}$$

where

$$\epsilon = [\epsilon_{11} \ 2\epsilon_{12} \ \epsilon_{22}]^T \quad \kappa = [\kappa_{11} \ 2\kappa_{12} \ \kappa_{22}]^T \tag{34}$$

$$\mathcal{A} = \langle D_{\parallel} \rangle \quad \mathcal{B} = \langle x_3 D_{\parallel} \rangle \quad \mathcal{D} = \langle x_3^2 D_{\parallel} \rangle \tag{35}$$

and

$$D_{\parallel} = D_e - D_{es}D_s^{*-T}D_{es}^{*T} - D_{et}D_t^{-T}D_{et}^{*T} \tag{36}$$

The strain energy per unit area including the first-order correction from initial twist can be expressed as

$$\begin{aligned}
2U_1 &= \langle \Gamma^{0T} D_1 \Gamma^0 - 2x_3 H \Gamma^{0T} D_0 \Gamma^0 + 2\Gamma^{1T} D_0 \Gamma^0 \rangle \\
&= \begin{Bmatrix} \epsilon \\ \kappa \end{Bmatrix}^T \begin{bmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{B}_1^T & \mathcal{D}_1 \end{bmatrix} \begin{Bmatrix} \epsilon \\ \kappa \end{Bmatrix}
\end{aligned} \tag{37}$$

Till now, we have successfully reduced the original 3-D strain energy to a 2-D shell strain energy which has an accuracy asymptotically correct through the first order of the initial twist  $k_1$ .

## Dimensional Reduction from 2-D to 1-D

The previously obtained model ion Eqs. (33) and (37) is an asymptotically correct classical shell model with geometric correction through the first order due to initial curvature of the reference surface. However, engineering practice often requires a more simplified model to carry out the relevant analysis and design of thin-walled beams. We need to proceed further to reduce the shell model to a 1-D beam model.

The deformed reference surface can be expressed in terms of beam quantities such that:

$$\mathbf{R}(x_1, x_2) = \mathbf{R}_0(y_1) + y_a(x_2)\mathbf{B}_a(y_1) + v_i(y_1, y_2, y_3)\mathbf{B}_i(y_1) \quad (38)$$

where  $\mathbf{B}_i$  are the unit vectors associated with  $y_i$  for the deformed configuration, and  $v_i$  is the warping field subject to the constraints

$$\langle\langle v_i \rangle\rangle = 0 \quad \langle\langle y_3 v_2 - y_2 v_3 \rangle\rangle = 0 \quad (39)$$

with the double angle-brackets denoting the definite integral along the contour of beam sections.

From Eq. (38), one can calculate the base vectors of the deformed shell surface  $\mathbf{A}_i$  based on their definitions, Eq. (13). Then one can obtain the fundamental forms of this surface from Eq. (20) and finally derive the shell strain measures in terms of beam quantities and warping functions from Eqs. (19). By neglecting all nonlinear terms with respect to the beam strain measures and warping functions  $v_i$ , one can find the 2-D shell strain measures asymptotically correct through the first order of initial twist as

$$\begin{aligned} \epsilon_{11} &= \gamma_{11} + \kappa_2 y_3 - \kappa_3 y_2 + k_1 \kappa_1 R^2 \\ 2\epsilon_{12} &= \dot{v}_1 + \kappa_1 R_n + k_1 (y_2 \dot{v}_3 - y_3 \dot{v}_2 + v_2 \dot{y}_3 - v_3 \dot{y}_2) \\ \epsilon_{22} &= \dot{y}_a \dot{v}_a \\ \kappa_{11} &= \kappa_a \dot{y}_a + \frac{1}{2} k_1 (\dot{v}_1 - 3\kappa_1 R_n) \\ 2\kappa_{12} &= -2\kappa_1 + \frac{b_{22}}{2} (\dot{v}_1 + \kappa_1 R_n) \\ &\quad + k_1 [2R_\tau (\kappa_2 \dot{y}_3 - \kappa_3 \dot{y}_2) - \dot{y}_a \dot{v}_a + \dot{R}_\tau (\gamma_{11} + \kappa_2 y_3 - \kappa_3 y_2) \\ &\quad + \frac{b_{22}}{2} (y_2 \dot{v}_3 - y_3 \dot{v}_2 + v_2 \dot{y}_3 - v_3 \dot{y}_2)] \\ \kappa_{22} &= (\dot{y}_3 \dot{v}_2 - \dot{y}_2 \dot{v}_3)_{,2} + k_1 [\kappa_1 R_\tau^2 b_{22} + R_\tau \ddot{v}_1 + \frac{\dot{R}_\tau}{2} (\dot{v}_1 + \kappa_1 R_n)] \end{aligned} \quad (40)$$

where  $\gamma_{11}$  is the extensional strain and  $\kappa_1$  the torsional strain and  $\kappa_a$  bending strains in the

$y_a$  direction.

To obtain the strain energy defined along the beam axis through the first order of initial twist, one needs to solve for the  $v_i$  of the zeroth-order approximation. Using the 2-D strain field in Eqs. (40) without the terms related with  $k_1$  and substituting it into the zeroth-order shell energy, Eq. (33), one can solve for the warping functions subject to the constraints in Eqs. (39). For the convenience of calculation, one can express the zeroth-order 2-D shell strains in matrix form as

$$\begin{Bmatrix} \epsilon \\ \kappa \end{Bmatrix} = P\varepsilon + T\psi \quad (41)$$

with

$$P = \begin{bmatrix} 1 & 0 & y_3 & -y_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \dot{y}_2 & \dot{y}_3 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \frac{b_{22}}{2} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\varepsilon = [\gamma_{11} \ \kappa_1 \ \kappa_2 \ \kappa_3]^T$$

$$\psi = [2\epsilon_{12}^0 \ \epsilon_{22}^0 \ \kappa_{22}^0]^T \quad (42)$$

The only unknowns exist in  $\psi$  and the constraints in Eqs. (39) are only needed to recover  $v_i$  and have no effect on the arbitrariness of  $\psi$ . Denoting the stiffness obtained in Eq. (33) as  $K$ , one can express the zeroth order energy per unit length of the beam axis using Eq. (41) as

$$2\Pi_0 = \langle\langle (P\varepsilon + T\psi)^T K (P\varepsilon + T\psi) \rangle\rangle \quad (43)$$

For open sections, there are no additional constraints on  $\psi$ . The minimization problem can be carried out in a straightforward manner, yielding

$$\psi = -(T^T K T)^{-1} T^T K P \varepsilon \quad (44)$$

Then the zeroth order energy can be expressed in terms of the beam strains as

$$2\Pi_0 = \varepsilon^T \langle\langle P^T [K - KT(T^T KT)^{-1}T^T K] P \rangle\rangle \varepsilon \quad (45)$$

The zeroth-order warping functions can be solved from Eq. (44) along with Eqs. (39). Substituting the obtained warping functions into Eq. (40) and then into Eq. (37), one should be able to obtain the strain energy per unit length expressed in terms of beam strain measures of the order  $k_1$ .

For closed sections, four additional constraints should be applied to ensure the uniqueness of the displacement field [9]. They are

$$\langle\langle v_{i,2} \rangle\rangle = 0 \quad \langle\langle \kappa_{22}^0 \rangle\rangle = 0 \quad (46)$$

These constraints can be transformed into the matrix form

$$\langle\langle \phi\psi - L\varepsilon \rangle\rangle = 0 \quad (47)$$

with

$$\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dot{y}_2 & -y_3 \\ 0 & \dot{y}_3 & y_2 \\ 0 & 0 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 0 & R_n & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (48)$$

Introducing Lagrange multipliers, the functional to be minimized has the form

$$2\Lambda = \langle\langle (P\varepsilon + T\psi)^T K (P\varepsilon + T\psi) + 2\lambda^T \phi\psi \rangle\rangle \quad (49)$$

where one can solve for  $\psi$  as

$$\psi = -(T^T KT)^{-1}(T^T KP\varepsilon + \phi^T \lambda) \quad (50)$$

Substituting the above equation into the constraints in Eq. (47), one can solve for  $\lambda$  as

$$\lambda = - \langle \langle \phi (T^T K T)^{-1} \phi^T \rangle \rangle^{-1} \langle \langle \phi (T^T K T)^{-1} T^T K P + L \rangle \rangle \varepsilon \quad (51)$$

Substituting Eq. (51) back into Eq. (50), one obtains  $\psi$  and then recovers the zeroth-order warping  $v_i$  with the help of Eq. (39). Finally one obtains the strain energy per unit length asymptotically correct through the first order of initial twist  $k_1$  from Eqs. (33) and (37).

It is worthy to emphasize that the present theory asymptotically reduces the original 3-D energy into a 1-D energy similar to that of classical beam theory. Hence the 1-D beam analysis remains essentially the same including the governing differential equations and boundary conditions as long as the beam analysis uses 1-D strain measures equivalent to those of the present theory.

## Numerical Examples

To demonstrate the usage and accuracy of the present theory, we study some simple examples such as an isotropic strip, a composite strip, an isotropic box-beam, and a composite box-beam.

The first example is a strip with width  $c$ , thickness  $h$  and initial twist  $k_1$ . This strip is made with isotropic material with Young's modulus as  $E$  and Poisson's ratio  $\nu$ . The warping functions in Eq. (12) can be solved according to Eq. (30) yielding

$$\begin{aligned} w_\alpha &= 0 \\ w_3 &= \frac{\nu}{\nu - 1} \left[ x_3(\epsilon_{11} + \epsilon_{22}) + \left( \frac{x_3^2}{2} - \frac{h^2}{24} \right) (\kappa_{11} + \kappa_{22}) \right] \end{aligned} \quad (52)$$

The warping functions in Eq. (38) can be solved according to Eq. (44), yielding

$$\begin{aligned} v_1 &= 0 \\ v_2 &= \frac{1}{2} \left( y_2^2 - \frac{c^2}{12} \right) \kappa_3 \nu - y_2 \gamma_{11} \nu \\ v_3 &= \frac{1}{2} \left( y_2^2 - \frac{c^2}{12} \right) \kappa_2 \nu \end{aligned} \quad (53)$$

Substituting the above warping functions into the 2-D shell strain measures in Eq. (40), one can obtain the strain measures for the strip with the first-order correction of initial twist

expressed in terms of beam strains as

$$\begin{aligned}
\epsilon_{11} &= \gamma_{11} - \kappa_3 y_2 + k_1 \kappa_1 y_2^2 \\
2\epsilon_{12} &= \frac{1}{2} k_1 \left( y_2^2 + \frac{c^2}{12} \right) \kappa_2 \nu \\
\epsilon_{22} &= y_2 \kappa_3 \nu - \gamma_{11} \nu \\
\kappa_{11} &= \kappa_2 \\
2\kappa_{12} &= -2\kappa_1 + k_1 \gamma_{11} (\nu + 1) - k_1 (3 + \nu) y_2 \kappa_3 \\
\kappa_{22} &= -\kappa_2 \nu
\end{aligned} \tag{54}$$

Substituting Eqs. (54) into Eqs. (33) and (37), one can calculate beam stiffness with first-order correction of  $k_1$  as

$$\begin{aligned}
S_{11} &= Ech \\
S_{12} &= \frac{Ec^3 h}{12} \left[ 1 - 3 \left( \frac{h}{c} \right)^2 \right] k_1 \\
S_{22} &= \frac{Ech^3}{6(\nu + 1)} \\
S_{33} &= \frac{Ech^3}{12} \\
S_{44} &= \frac{Ec^3 h}{12}
\end{aligned} \tag{55}$$

where  $S_{11}$  is the extensional stiffness,  $S_{22}$  torsional stiffness,  $S_{33}$  bending stiffness in the  $x^2$  direction,  $S_{44}$  bending stiffness in the  $x^3$  direction, and  $S_{12}$  is the extension-twist coupling from initial twist. This value is the same as that for an initially twisted isotropic solid beam [15]

$$\begin{aligned}
S_{12} &= [S_{33} + S_{44} - 2(\nu + 1)S_{22}]k_1 \\
&= \frac{Ec^3 h}{12} \left[ 1 - 3 \left( \frac{h}{c} \right)^2 \right] k_1
\end{aligned} \tag{56}$$

Eq. (56) is derived using the VAM without taking advantage of the smallness of wall thickness and is asymptotically correct though the first order of the initial twist. The value provided in Ref. [16] was

$$S_{12} = \frac{Ec^3 h}{12} \left[ 1 - \left( \frac{h}{c} \right)^2 \right] k_1 \tag{57}$$

Although both results converge to the same value as  $h/c$  goes to zero, the result of Ref. [16]

introduces an error of the order  $(h/c)^2$  in comparison to that of Ref. [15], which is inevitable when one attempts to take advantage of the smallness of wall thickness in an *ad hoc* manner that is not asymptotically correct. On the other hand, the present theory, which is still a thin-walled beam theory, can reproduce the asymptotically correct results obtained without invoking the thin-walled assumption for this particular case.

The second example is a single-layer strip made from an anisotropic material with properties given by

$$\begin{aligned}
 E_{11} &= 25 \times 10^6 \text{ psi} & E_{22} &= E_{33} = 10 \times 10^6 \text{ psi} \\
 G_{12} &= G_{13} = 5 \times 10^6 \text{ psi} & G_{23} &= 2 \times 10^6 \text{ psi} \\
 \nu_{12} &= \nu_{13} = \nu_{23} = 0.25 & & 
 \end{aligned} \tag{58}$$

The ply orientation is  $-15^\circ$ . Even if there is no initial twist, this strip exhibits bending-twist coupling. We are going to compare our result with VABS which has been comprehensively validated against experimental results, 3-D elasticity solutions, and 3-D finite element solutions [11, 17–20]. For the purpose of comparing the results with VABS, we assume  $c = 2$  in.,  $h = 0.2$  in., and  $k_1 = 0.1/\text{in.}$  The results were tabulated in Table 1. One can observe that the present theory, which takes advantage of the thin-walled structure, agrees reasonably well with VABS, which is also based on the variational asymptotic method but is capable of treating cross sections of arbitrary geometry. The present theory gives the exact solution for the stiffness in the stiff directions such as extension and bending around the normal of the strip. However, for stiffness in the soft directions, particularly the torsional stiffness and bending-twist coupling, the present theory introduces some errors because of the fact we have taken advantage of the smallness of  $h/c$  during the reduction from 3-D to 2-D.

When the strip is initially twisted, extension-twisting coupling will appear in the stiffness model. The present theory yields an excellent prediction for the extension-twisting coupling. The present theory also predicts a very weak extension-bending coupling which is three orders of magnitude less than the extension-twisting coupling. Although VABS also predicts such a small term, it is not meaningful to judge the accuracy based on this difference. It is not clear to the authors at this stage what would be the source of this term, although the values are reported for completeness.

It may seem that the errors for  $S_{22}$  and  $S_{23}$  are too large to be acceptable. It is comforting to find out that even for an untwisted isotropic strip with the same aspect ratio ( $c/h = 10$ ) the exact torsional stiffness ( $0.312Gch^3$ ) is about 6.7% different from the result  $Gch^3/3$  that is based on thin-walled assumptions. Therefore, the amount of error for composite beams as shown in Table 1 is not unreasonable. Theories such as the present one, which

take advantage of the smallness of thickness, should converge to the exact solution when the aspect ratio  $h/c$  goes to zero. For this purpose, initially twisted composite strips with different aspect ratios are studied; the convergence trends of  $S_{12}$ ,  $S_{22}$  are plotted in Fig. 2 where the dashed line-dot plot for  $S_{12}$  and solid line-dot for  $S_{22}$ . The values of the present theory are normalized by VABS results. It is found that that  $S_{23}$  always has the same ratio as  $S_{22}$ , which can be explained by the fact that both values are proportional to  $ch^3$ .  $S_{12}$  converges to the VABS result faster than  $S_{22}$  because part of it is contributed by the bending stiffness in the stiff direction according to Eq. (56). The torsional stiffness is very sensitive to the assumptions one uses to develop the theory. Even when the aspect ratio is 30, the difference of the torsional stiffness  $S_{22}$  has a value around 2.3%. Nevertheless, as one can observe from the plot, the values from the present theory indeed converge to the more accurate values produced by VABS.

The third example is a thin-walled isotropic box-beam with length  $a$ , width  $b$ , and thickness  $h$  (see Fig. 3 for dimensions). The box-beam is made of isotropic material with Young's modulus  $E$  and Poission's ratio  $\nu$ . Following a similar procedure as for the isotropic strip, analytical formulas can be obtained for the stiffnesses as

$$\begin{aligned}
S_{11} &= 2(a+b)Eh \\
S_{12} &= \frac{Ehk_1(a^4 + 4ba^3 - 3a^2(2b^2 + h^2) + a(4b^3 - 6bh^2) + b^4 - 3b^2h^2)}{6(a+b)} \\
S_{22} &= \frac{Eh(a^2(3b^2 + h^2) + 2abh^2 + b^2h^2)}{3(a+b)(1+\nu)} \\
S_{33} &= \frac{Eh(b^3 + 3ab^2 + ah^2)}{6} \\
S_{44} &= \frac{Eh(a^3 + 3a^2b + bh^2)}{6}
\end{aligned} \tag{59}$$

One can verify that the extension-twisting coupling satisfies the exact relation in Eq. (56). Note that  $S_{11}$ ,  $S_{33}$ , and  $S_{44}$  are the same as the exact solutions. The difference between the traditional thin-walled theory (assuming  $h/a$  and  $h/b$  are zero) and the present theory is of the order of  $(h/c)^2$ , with  $c$  denoting the larger value of  $a$  and  $b$ .

The last example is a pretwisted thin-walled box-beam with length  $a = 0.923$  in., width  $b = 0.5$  in., thickness  $h = 0.03$  in., and initial twist  $k_1 = 0.1$ /in. This box-beam is made of an anisotropic material with the same properties as those for the composite strip, which are given by Eq. (58). The six layers have the same ply angles of  $-15^\circ$ . The layup angle is positive if the fiber rotates from the beam axis around the outward normal of each wall counterclockwisely. The results are shown in Table 2. Again excellent agreement is found between VABS and the present theory. Except for the torsional stiffness, the differences

of other values are less than 1%. It is noticed that without including  $D_1$  in Eq. (25) an error of about 20% is introduced for  $S_{12}$  in comparison to VABS results. Hence a systematic dimensional reduction should be carried out by consistently including all the first order correction due to  $k_1$  in both steps of reduction from 3-D to 2-D and from 2-D to 1-D. Direct use of stiffness matrices from classical shell theory without geometric correction, as suggested in Refs. [9,21], will introduce some differences to the final thin-walled beam models with initial twist. However, for the isotropic case it is verified that differences between the present theory and those of Refs. [9,21] are of the order of  $(h/c)^2$ , which is allowable in thin-walled beam theories (i.e. theories that take advantage of the smallness of  $h/c$ ). Indeed, it is encouraging to find out that results from the present theory satisfy the analytical formula in Eq. (56), obtained using the VAM for arbitrary beams. In spite of the discontinuity in the slope of the contour, the box-beam case clearly demonstrates the usage and accuracy of the present theory for thin-walled beams with closed sections.

## Conclusions

A general framework to model initially twisted, thin-walled, composite beams has been developed to consistently capture the first-order correction to the beam stiffness from initial twist. The unique geometry of a thin-walled beam ( $h/c \ll 1$  and  $c/l \ll 1$ ) makes it possible to analytically reduce the original 3-D representation to a shell model and then further to a beam model. It is shown that to consistently calculate the first-order correction from initial twist one has to obtain this correction during both dimensional reductions, which is against the usual practice in the literature that the pretwisted thin-walled beams are modeled directly from a classical shell theory without geometric corrections.

Simple strip and box-beam examples have been studied to demonstrate the usage and accuracy of the present theory. Results show that the present theory provides a very accurate beam model to approximate the original 3-D representation. The present theory can be used with confidence in the design and analysis of pretwisted thin-walled composite beams. A good compromise between accuracy and efficiency has been achieved for modeling thin-walled beams so that tradeoffs and analysis can be performed efficiently and without significant loss of accuracy. It is worthy to point out that the present theory applies to all types of thin-walled beams including strips and closed or open sections without using different theory for different types as traditional techniques demand.

The analytical formulas developed here are implemented using Mathematica<sup>TM</sup>, a symbolic manipulator. The present theory provides an alternative to VABS for structure engineers who wants a quick way to calculate the section properties of composite thin-walled beams without tedious modeling effort and computation cost of the finite element method.

Any single-celled, thin-walled cross section can be modeled using the present theory. Additional work is required to extend the present theory to include multiple cells or to introduce refinements to the 1-D model such as shear deformation or the Vlasov effect, and corrections for initial curvature and higher-order corrections, which are already treated by VABS.

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## **List of Table Captions**

**Table 1:** Stiffness for Pretwisted Composite Strip

**Table 2:** Stiffness for Pretwisted Composite Box-Beam

$S$	VABS	Present	Difference
$S_{11}$	$0.85985931 \times 10^7$	$0.85985931 \times 10^7$	0%
$S_{12}$	$0.27792311 \times 10^6$	$0.28043542 \times 10^6$	0.9%
$S_{22}$	$0.27868182 \times 10^5$	$0.29968253 \times 10^5$	7.5%
$S_{13}$	$0.24939907 \times 10^2$	$0.61395754 \times 10^3$	
$S_{23}$	$0.70870494 \times 10^4$	$0.76211033 \times 10^4$	7.5%
$S_{33}$	$0.30464257 \times 10^5$	$0.30600072 \times 10^5$	0.5%
$S_{44}$	$0.28661977 \times 10^7$	$0.28661977 \times 10^7$	0%

TABLE 1:

$S$	VABS	Present	Difference
$S_{11}$	$0.19600322 \times 10^7$	$0.19594756 \times 10^7$	0.028%
$S_{12}$	$0.91355458 \times 10^5$	$0.91558325 \times 10^5$	0.22%
$S_{22}$	$0.52561275 \times 10^5$	$0.51607671 \times 10^5$	1.8%
$S_{33}$	$0.90793318 \times 10^5$	$0.90459553 \times 10^5$	0.37%
$S_{44}$	$0.23141874 \times 10^6$	$0.23051801 \times 10^6$	0.39%

TABLE 2:

## List of Figure Captions

**Figure 1: Schematic of A Thin-walled Beam**

**Figure 2: Convergence Study of Thin-walled Beam Theory**

**Figure 3: Schematic of Cross-section of A Thin-walled Box-beam**

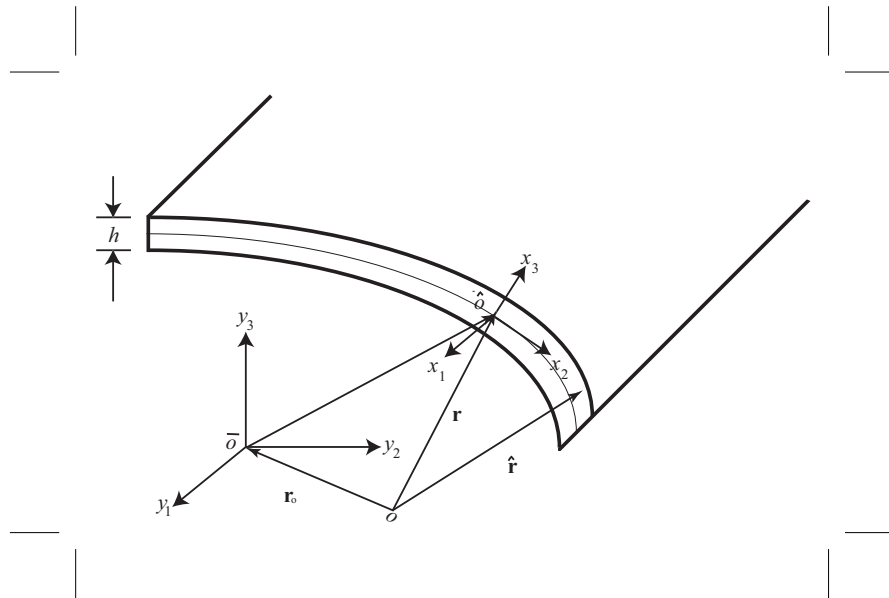


FIGURE 1:

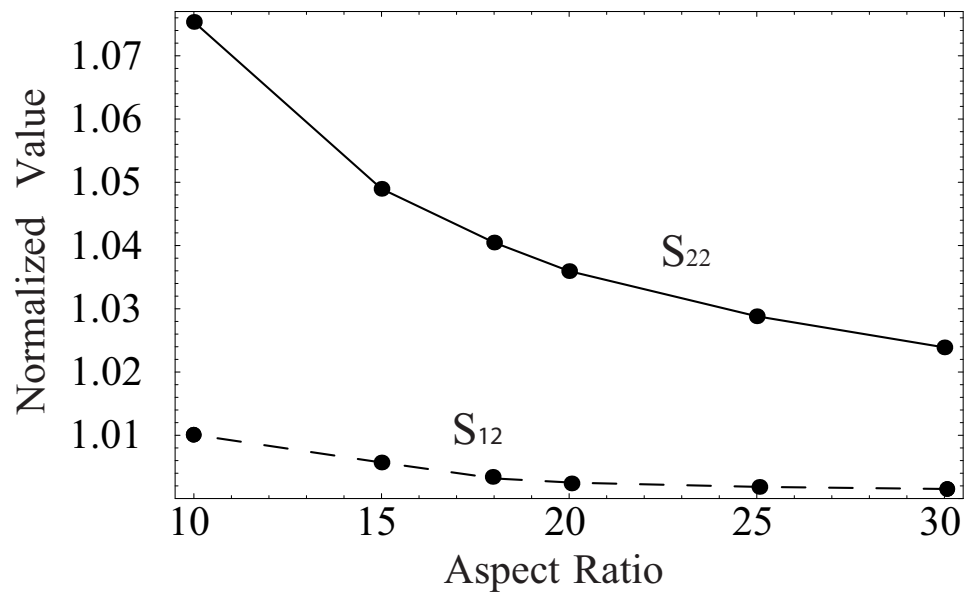


FIGURE 2:

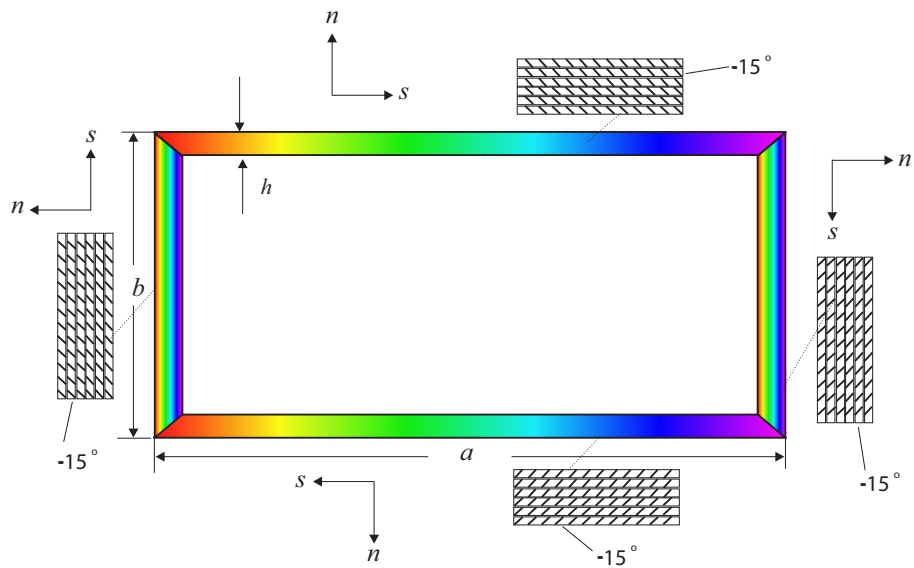


FIGURE 3: